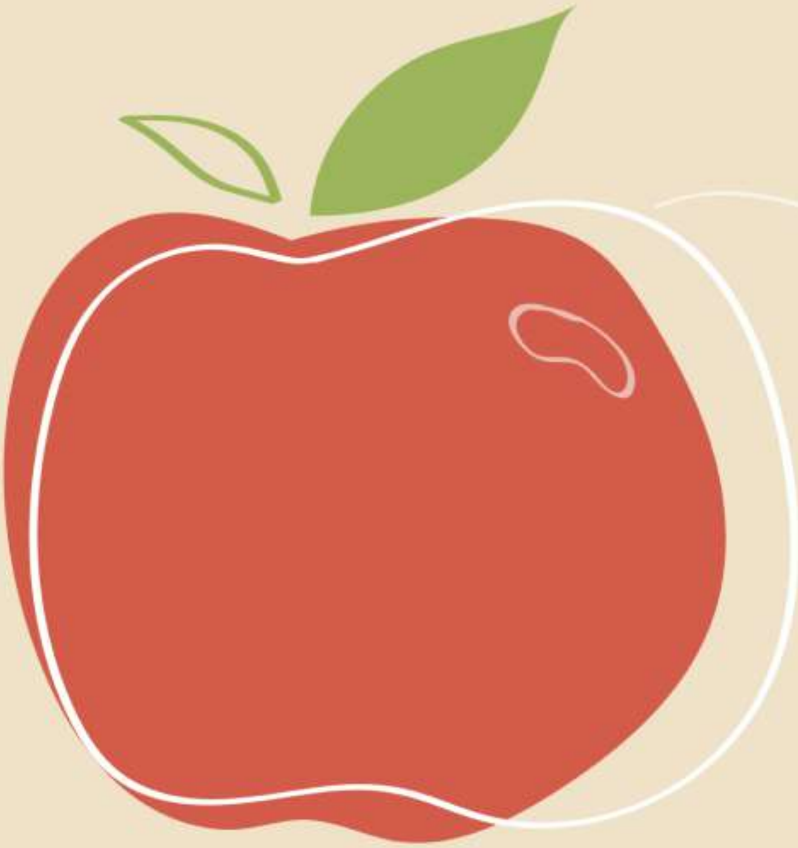
MA 412
complex analysis



30% – HW
10% – Quiz
25% – midsem
35% – endsem
5% – show up for Quiz, 5% – choose problem
Quiz – Redo possible
Show up initially

7th Jan:

set of complex numbers is denoted by $\mathbb{C} = \{x+iy \mid x, y \in \mathbb{R}\}$ ← field of real numbers

set → Field (complex number is a field)

Note: $i^2 = -1$

Construction of $\mathbb{C}$: Consider $\mathbb{R}\times\mathbb{R}$ is set of 2-tuples ← direct product

$$\mathbb{R}\times\mathbb{R} = \{(a,b) \mid a, b \in \mathbb{R}\}$$

equipped with $+$ & $\cdot$

(a_1, b_1) & $(a_2, b_2) \in \mathbb{R}\times\mathbb{R}$, define $+$ by

$$(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$$

Also observe $(0,0)$ is additive identity.

ie $(a,b) + (0,0) = (0,0) + (a,b) = (a,b)$

Defination of Product $(\cdot)$:

$$(a_1, b_1)\cdot(a_2, b_2) = (a_1a_2 - b_1b_2,\ a_1b_2 + a_2b_1)$$

Check: ① Multiplication is commutative

$$(a_1, b_1)\cdot(a_2, b_2) = (a_2, b_2)\cdot(a_1, b_1)$$

② Associativity: $[(a_1,b_1)\cdot(a_2,b_2)]\cdot(a_3,b_3) = (a_1,b_1)\cdot[(a_2,b_2)\cdot(a_3,b_3)]$

③ Distributivity of $\cdot$ over $+$:

$$(a_1, b_1)\cdot[(a_2, b_2) + (a_3, b_3)]$$
$$= (a_1, b_1)\cdot(a_2, b_2) + (a_1, b_1)\cdot(a_3, b_3)$$

proof:

① commutative proof:

$$(a_1, b_1)\cdot(a_2, b_2) = (a_1a_2 - b_1b_2,\ a_1b_2 + b_1a_2)$$
$$(a_2, b_2)\cdot(a_1, b_1) = (a_2a_1 - b_2b_1,\ a_1b_2 + b_1a_2)$$
$$= (a_1a_2 - b_1b_2,\ a_1b_2 + b_1a_2)$$
$$= RHS$$

② $[(a_1, b_1)\cdot(a_2, b_2)]\cdot(a_3, b_3)$

$$= [(a_1a_2 - b_1b_2,\ a_1b_2 + b_1a_2)]\cdot(a_3, b_3)$$
$$= (\cancel{a_1a_2a_3} - \cancel{b_1b_2a_3} - a_1b_2b_3 - b_1a_2b_3,$$
$$\cancel{a_1a_2b_3} - \cancel{b_1b_2b_3} + \cancel{a_1a_3b_2} + \cancel{b_1a_2a_3})$$

and

$$(a_1, b_1)\cdot[(a_2, b_2)\cdot(a_3, b_3)]$$
$$= (a_1, b_1)\cdot[(a_2a_3 - b_2b_3,\ a_2b_3 + a_3b_2)]$$
$$= (\cancel{a_1a_2a_3} - a_1b_2b_3 - b_1a_2b_3 - b_1a_3b_2,$$
$$\cancel{a_1a_2b_3} + \cancel{a_1a_3b_2} + \cancel{a_2a_3b_1} - \cancel{b_1b_2b_3})$$

LHS = RHS

③ $(a_1, b_1)\cdot[(a_2, b_2) + (a_3, b_3)]$

$$= (a_1, b_1)\cdot[(a_2 + a_3, 0) + (0, b_2 + b_3)]$$

$$= (a_1a_2 + a_1a_3,\ b_1a_2 + b_1a_3)$$
$$+ (-b_1b_2 - b_1b_3,\ a_1b_2 + a_1b_3)$$
$$= a_1b_1(a_2+a_3,\ a_2+a_3)$$
$$- b_1a_1(b_2+b_3,\ b_2+b_3)$$
$$= a_1b_1[(a_2+a_3-b_2-b_3,\ a_2+a_3-b_2-b_3)]$$

$(a_1,b_1)\cdot(a_2,b_2)$
$+ (a_1,b_1)\cdot(a_3,b_3)$
$$= (a_1a_2, a_2b_1) + (-b_1b_2, a_1b_2)$$
$$+ (a_1a_3, a_3b_1) + (-b_1b_3, a_1b_3)$$
$$= a_1b_1[(a_2+a_3,\ a_2+a_3) - (b_2+b_3,\ b_2+b_3)]$$
LHS = RHS

Ex: $(a,b) = (a,0) + (0,b)$

Here this can help make calculations

$(0,1)\cdot(0,1) = (-1,0)$

$\mathbb{R} \hookrightarrow \{\mathbb{R}\times\mathbb{R}, +, \cdot\}$
$a \longmapsto (a,0)$

check: ④ There are no zero divisors
$(a_1,b_1)\cdot(a_2,b_2) = (0,0) \Leftrightarrow (a_1,b_1)=0$
or
$(a_2,b_2)=0$

⑤ If $(a,b)\neq(0,0)$, then $\exists\ (c,d)$ s.t $(a,b)\cdot(c,d) = (1,0)$

proof: ④ $(\Rightarrow)$ $(a_1,b_1)\cdot(a_2,b_2) = (0,0)$
then
$$(a_1a_2 - b_1b_2,\ a_1b_2 + a_2b_1) = (0,0)$$
$$\Rightarrow a_1a_2 = b_1b_2$$
$$\& \ a_1b_2 = -a_2b_1$$

now if $(a_1,b_1)\neq 0$
$\&$ $(a_2,b_2)\neq 0$
then
$a_2 = \frac{-b_1b_2}{a_1} \Rightarrow a_1\cancel{b_2} = \frac{-b_1\cancel{b_2}}{a_1}b_1$ (wlog $a_1\neq 0$)

$$\Rightarrow a_1^2 = -b_1^2$$
as $a_1^2 = -b_1^2$
and both
$a_1^2, b_1^2 > 0$
this is a contradiction
so $(a_1,b_1)=0$ or $(a_2,b_2)=0$

$(\Leftarrow)$ wlog $(a_1,b_1)=0$ then
$(0,0)\cdot(a_2,b_2) = (0,0) = 0$

⑤ if $(a,b) \neq 0$ then
wlog $a \neq 0$ and now
$(a, b)\cdot(c,d) = (1,0)$

$(ac-bd, bc+ad) = (1,0)$
$ac-bd=1$
$bc+ad=0$
$ac = 1+bd$
$ad = -bc$
$d = -\frac{bc}{a}$
$ac = 1+b\left(-\frac{bc}{a}\right)$
$ac = 1-\frac{b^2c}{a}$
$a^2c = a-b^2c$
$c = \frac{a}{a^2+b^2}$
$d = -\frac{ba}{a^2+b^2}$

Note: $(a,b) = a+ib$

Construction – II :

$\mathbb{R}$ – field
$\mathbb{R}[x]$ = ring of polynomials over $\mathbb{R}$
→ $\mathbb{R}[x]$ is a PID (principal ideal domain)
maximal ideals ⇔ prime ideals ⇔ irreducible elements

consider $\frac{\mathbb{R}[x]}{(x^2+1)}$
Here (x^2+1) is maximal ⇒ $\frac{\mathbb{R}[x]}{(x^2+1)}$ is a field
so $\mathbb{R}\times\mathbb{R}$ is also a field ($\mathbb{C}$)

To prove: $\frac{\mathbb{R}[x]}{(x^2+1)} \longrightarrow \{\mathbb{R}\times\mathbb{R}, +, \cdot\}$ is a field isomorphism

proof: $a+bx \in \frac{\mathbb{R}[x]}{(x^2+1)}$
$F : \frac{\mathbb{R}[x]}{(x^2+1)} \longrightarrow \{\mathbb{R}\times\mathbb{R}, +, \cdot\}$
$ax+b \longmapsto (a,b)$, trivially isomorphism
(homomorphism + one-one + onto)
$F(ax+b) = (a,b)$

① — $F\left[\alpha(ax+b)+(cx+d)\right] = (\alpha a+c, \alpha b+d)$
$= \alpha F(ax+b) + F(cx+d)$

$F(ax+b) = F(cx+d)$
$a=c,\ b=d$
$\Rightarrow ax+b = cx+d$

$\forall (a,b)\in \mathbb{R}\times\mathbb{R}$, $\exists\ ax+b \in \frac{\mathbb{R}[x]}{(x^2+1)}$ s.t $F(ax+b) = (a,b)$

Construction - III :

$$\left\{ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \middle| \alpha, \beta \in \mathbb{R} \right\}$$ with regular matrix + ρ x is isomorphic to $\mathbb{C}$.

Notation :

For $z \in \mathbb{C}$, write $z = x + iy$ for $x, y \in \mathbb{R}$

$x = Re(z)$ and

$y = im(z)$

Complex conjugate of z is $\bar{z} = x - iy$

Absolute values :

$$|z|^2 = z \cdot \bar{z} \text{ s.t } |z| \geq 0$$

$$|z|^2 = (x+iy)(x-iy) = x^2 + y^2$$

Ex : compute $(1+2i)^3$

Ans :

$$(1+2i)^3 = (1,2)\cdot(1,2)\cdot(1,2) = (1,2)(1-4, 2+2)$$
$$= (1,2)(-3,4)$$
$$= (-3-8, -6+4)$$
$$= (-11, -2) = -11 - 2i$$

Properties of conjugation & |·|.

1) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

2) $\overline{z_1 . z_2} = \bar{z}_1 \cdot \bar{z}_2$ (By $z_j = a_j + i b_j$ and computing)

3) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ write

Here $|z_1 \cdot z_2| = |(a_1 + ib_1)\cdot(a_2 + ib_2)|$
$= |(a_1^2 + b_1^2)^{1/2}(a_2^2 + b_2^2)^{1/2}|$

one more way :

$$|z_1 \cdot z_2|^2 = (z_1 \cdot z_2)\cdot(\overline{z_1 \cdot z_2})$$
$$= (z_1 \bar{z}_1)(z_2 \bar{z}_2)$$
$$= |(z_1)|^2 |(z_2)|^2$$
$$|z_1 z_2|^2 = |z_1|^2 |z_2|^2$$
$$\Rightarrow |z_1 z_2| = |z_1||z_2|$$

Triangle inequality : For $z_1, z_2 \in \mathbb{C}$

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

proof : Square LHS

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2})$$
$$= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2)$$
$$= (z_1 \bar{z}_1) + (z_2 \bar{z}_2) + z_1 \bar{z}_2 + \underbrace{z_2 \bar{z}_1}_{\overline{z_1 \bar{z}_2}}$$

$$= |z_1|^2 + |z_2|^2 + 2Re(z_1 \bar{z}_2)$$

for any $w \in \mathbb{C}$, we have $Re(w) \leq |w|$

so

$$|z_1|^2 + |z_2|^2 + 2Re(z_1 \bar{z}_2) \leq |z_1|^2 + |z_2|^2 + 2|z_1 \bar{z}_2|$$
$$= |z_1|^2 + |z_2|^2 + 2|z_1||z_2|$$
$$= (|z_1| + |z_2|)^2$$

Note : $|\bar{z}_1| = |z_1|$

$$-|w| \leq Re(w) \leq |w|$$

also

$$|z_1 + z_2| \leq |z_1| + |z_2|$$

but when is $|z_1| + |z_2| = |z_1 + z_2|$

when $Re(\bar{z}_1 z_2) = |\bar{z}_1 z_2|$

so $Im(\bar{z}_1 z_2) = 0$

and also $Re(\bar{z}_1 z_2) \geq 0$

also if $Im(\bar{z}_1 z_2) = 0$ and $Re(\bar{z}_1 z_2) \geq 0$

$\Rightarrow Re(\bar{z}_1 z_2) = |\bar{z}_1 z_2|$

as

$|\bar{z}_1 z_2| = \sqrt{[Re(\bar{z}_1 z_2)]^2}$ as $Re(\bar{z}_1 z_2) \geq 0$

also if $Re(z_1 \bar{z}_2) \geq 0$

Now, $Re(z_1 \bar{z}_2) = Re\left(\frac{z_1}{z_2} \cdot \bar{z}_2 z_2\right)$ (assuming $z_2 \neq 0$)

$$= |z_2|^2 Re\left(\frac{z_1}{z_2}\right) = |z_1 \bar{z}_2|$$

so when $|z_2|^2 Re\left(\frac{z_1}{z_2}\right) \geq 0$

$\frac{z_1}{z_2}$ is a non-neg real number

more generally: $|z_1 + \cdots z_n| \leq |z_1| + \cdots |z_n|$

By induction, say $|z_1 + \cdots z_{n-1}| \leq |z_1| + \cdots |z_{n-1}|$

then

$$|\underbrace{z_1 + \cdots}_{w} \underbrace{z_n}_{z_n}| \leq |w| + |z_n| \leq |z_1| + \cdots |z_{n-1}| + |z_n|$$

problem: show that if $z_1 \ldots z_n \neq 0$, then $|z_1 + \cdots z_n| = |z_1| + \cdots + |z_n|$ iff each $\frac{z_j}{z_k}$ is a non-negative real number

Ans: $(\Rightarrow)$ $|z_1 + z_2 + \cdots z_n|^2 = (z_1 + \cdots z_n) \cdot (\overline{z_1 + \cdots z_n})$

$$= \sum_i \sum_j z_i \bar{z}_j$$

$$= \sum |z_i|^2 + \sum_{i<j} 2Re(z_i \bar{z}_j)$$

$$= \left(\sum |z_i|\right)^2$$

$$\Rightarrow \sum |z_i|^2 + \sum 2Re(z_i \bar{z}_j) = \sum |z_i|^2 + 2\sum_{i<j} |z_i \bar{z}_j|$$

$$\Rightarrow \sum \cancel{2} Re(z_i \bar{z}_j) = \cancel{2} \sum_{i<j} |z_i \bar{z}_j|$$

$$\Rightarrow \sum_{i<j} Re\left(\frac{z_i}{z_j}\right) = \sum \left|\frac{z_i}{z_j}\right|$$

so, $\frac{z_i}{z_j}$ is non-neg real number as

$Re\left(\frac{z_i}{z_j}\right) \leq \left|\frac{z_i}{z_j}\right|$ and here

inequality is equated so

$Re\left(\frac{z_i}{z_j}\right) = \frac{|z_i|}{|z_j|}$ $\forall i, j$ s.t $i < j$

also we can do same for $i > j$

($\Leftarrow$) now if $\frac{z_i}{z_j}$ is non-neg real

then $\frac{z_i}{z_j} = a + 0i$ where $a \geq 0$

$\Rightarrow Re\left(\frac{z_i}{z_j}\right) = a = \frac{z_i}{z_j}$

$\Rightarrow \sum_{i<j} Re\left(\frac{z_i}{z_j}\right) = \sum_{i<j} \left|\frac{z_i}{z_j}\right|$

$\Rightarrow \left|\sum z_i\right| = \sum |z_i|$

Cauchys inequality:

$$\left|\sum_{k=1}^{n} a_k b_k\right|^2 \leq \left(\sum_{k=1}^{n} |a_k|^2\right)\left(\sum_{k=1}^{n} |b_k|^2\right) \text{ where } a_k, b_k \in \mathbb{C}$$

proof: ① induction for $n=2$ or to show:

$$|a_1b_1 + a_2b_2|^2 \leq (|a_1|^2 + |a_2|^2)(|b_1|^2 + |b_2|^2)$$

now, $|a_1b_1 + a_2b_2|^2$

$= (a_1b_1 + a_2b_2)\overline{(a_1b_1 + a_2b_2)}$

$= (a_1b_1 + a_2b_2)(\overline{a_1b_1} + \overline{a_2b_2})$

$= a_1b_1\overline{a_1}\,\overline{b_1} + a_1b_1\overline{a_2}\,\overline{b_2} + a_2b_2\overline{a_2b_2} + a_2b_2\overline{a_1b_1}$

$= |a_1|^2|b_1|^2 + |a_2|^2|b_2|^2 + 2Re(a_1b_1\overline{a_2}\,\overline{b_2})$

$\leq |a_1|^2|b_1|^2 + |a_2|^2|b_2|^2 + 2|a_1b_1\overline{a_2}\,\overline{b_2}|$

where $2|a_1||b_1||a_2||b_2| \leq |a_1|^2|b_2|^2 + |a_2|^2|b_1|^2$ (AM $\geq$ GM)

$\Rightarrow \leq |a_1|^2|b_1|^2 + |a_2|^2|b_2|^2 + |a_1|^2|b_2|^2 + |a_2|^2|b_1|^2$

$\therefore$ it is true for $n=2$, now for general case

say true for $n=k-1$

for $n=k$

$$\left|\sum^{k-1} a_ib_i\right|^2 \leq \left(\sum^{k-1}|a_i|^2\right)\left(\sum^{k-1}|b_i|^2\right)$$

then

$$\left|\sum^{k} a_ib_i\right|^2 = \left|\sum^{k-1} a_ib_i + a_kb_k\right|^2 \quad \text{(triangle inequality)}$$

$$\leq \left|\sum^{k-1} a_ib_i\right|^2 + |a_kb_k|^2$$

$$\leq \left(\sum^{k-1}|a_i|^2\right)\left(\sum^{k-1}|b_i|^2\right) + |a_k|^2|b_k|^2$$

$$\leq \left(\sum^{k-1}|a_i|^2\right)\left(\sum^{k-1}|b_i|^2\right) + |a_k|^2\sum^{k}|b_i|^2 + \sum^{k}|a_i|^2|b_k|^2$$

$$= \left(\sum^{k}|a_i|^2\right)\left(\sum^{k}|b_i|^2\right)$$

so true for $n=k$

$\therefore$ By induction, correct

② let λ be some number in $\mathbb{C}$.
construct $$\sum_{k=1}^{n} |a_k - \lambda \overline{b_k}|^2 \geq 0$$

here $|a_k - \lambda \overline{b_k}|^2 = (a_k - \lambda \overline{b_k}) \cdot (\overline{a_k} - \overline{\lambda} b_k)$
$= |a_k|^2 + |\lambda|^2 |b_k|^2 - 2\,\mathrm{Re}(\overline{\lambda} a_k b_k)$

summing all:

$$\sum |a_k - \lambda \overline{b_k}|^2 = \sum |a_k|^2 + \sum |\lambda|^2 |b_k|^2 - \sum 2\,\mathrm{Re}(\overline{\lambda} a_k b_k) \geq 0$$

get:

$$\Rightarrow \sum |a_k|^2 + \sum |b_k|^2 |\lambda|^2 \geq \sum 2\,\mathrm{Re}(\overline{\lambda} a_k b_k)$$

choose $\lambda = \dfrac{\sum a_k b_k}{\sum |b_k|^2}$ to

$$\sum |a_k|^2 + \frac{\left(\sum a_k b_k\right)^2}{\sum |b_k|^2} \geq \sum 2\,\mathrm{Re}\left[\left(\frac{\sum a_k b_k}{\sum |b_k|^2}\right) \overline{a_k} \overline{b_k}\right]$$

$$\Rightarrow \left(\sum |a_k|^2\right)\left(\sum |b_k|^2\right) + \cancel{(\sum a_k b_k)^2} \geq \cancel{2}\, (\sum a_k b_k)^2$$

$$\Rightarrow \left(\sum |a_k|^2\right)\left(\sum |b_k|^2\right) \geq \left(\sum a_k b_k\right)^2$$

8^{th} Jan :

Triangle inequality :

For $z_1, z_2 \in \mathbb{C}$, $\triangle$-inequality says $|z_1+z_2| \leq |z_1| + |z_2|$ and this can be generalised to n by induction.

$|z_1 - z_2| \geq |z_1| - |z_2|$ as

$$z_1 = (z_1 - z_2) + (z_2)$$
$$\Rightarrow |z_1| = |(z_1 - z_2) + z_2|$$
$$\leq |z_1 - z_2| + |z_2|$$

so

$$|z_1 - z_2| \geq |z_1| - |z_2|$$

and sim $|z_1 - z_2| \geq |z_2| - |z_1|$

$$\Rightarrow \quad |z_1 - z_2| \geq \big||z_1| - |z_2|\big|$$

this can be very helpfull

Cauchy's inequality :

let $a_1, \dots a_n$ and $b_1 \dots b_n \in \mathbb{C}$ then

$$\left|\sum_1^n a_k b_k\right|^2 \leq \left(\sum_1^n |a_k|^2\right)\left(\sum_1^n |b_k|^2\right)$$

Consider, $\sum^n |a_k - \lambda \bar{b}_k|^2 \geq 0$

in particular, choose $\lambda = \dfrac{\sum a_k b_k}{\sum |b_k|^2}$

Exe : when do we have LHS = RHS

Ans : as $\sum |a_k - \lambda \bar{b}_k|^2 \geq 0$

if $\sum |a_k - \lambda \bar{b}_k|^2 = 0 \Rightarrow$ each

$$|a_i - \lambda \bar{b}_i|^2 = 0$$
$$\Leftrightarrow a_i = \lambda \bar{b}_i \quad \forall i = 1, 2 \dots n$$
$$\Leftrightarrow \frac{a_1}{\bar{b}_1} = \frac{a_2}{\bar{b}_2} = \frac{a_3}{\bar{b}_3} = \dots \frac{a_n}{\bar{b}_n}$$

we don't have to worry about any ai/bi being 0 as that term will vanish.

Exe : why do we choose this λ $\left(\lambda = \dfrac{\sum a_i b_i}{\sum |b_i|^2}\right)$

Ans : This is similar to calculus of multivariable

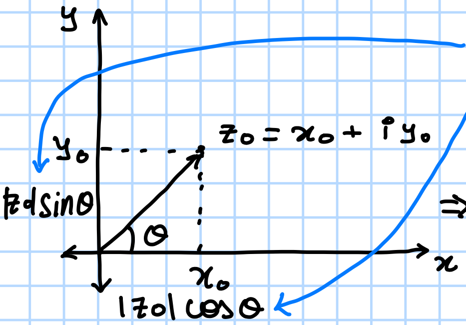
$$|a_k - \lambda \bar{b}_k|^2 = (a_k - \lambda \bar{b}_k)(\bar{a}_k - \bar{\lambda} b_k)$$
$$= |a_k|^2 - \bar{\lambda} a_k b_k - \lambda \overline{a_k b_k} + |\lambda|^2 |b_k|^2$$
$$= \underbrace{|a_k|^2 + |\lambda|^2 |b_k|^2 - 2\,\mathrm{Re}(\bar{\lambda} a_k b_k)}_{\geq 0}$$

$$\Rightarrow \sum |a_k|^2 + |\lambda|^2 \sum |b_k|^2 \geq 2 \sum \mathrm{Re}(\bar{\lambda} a_k b_k)$$

$$\cancel{2} |\lambda| \, |b_k|^2 = \cancel{2} \left|\sum a_k b_k\right|$$

$$\lambda = \frac{\sum a_k b_k}{|b_k|^2} \leftarrow \text{By calculus}$$

Complex plane :



As this work for 1/4 of the plane, it works for all plane's, but proof is needed

$$|z_0|^2 = z_0\bar{z}_0$$
$$\Rightarrow |z_0| = \sqrt{z_0\bar{z}_0}$$
$$= \sqrt{x_0^2 + y_0^2}$$ ← This is exactly like the usual distance

$$z_0 = x_0 + iy_0 \neq 0$$
$$= \sqrt{x_0^2+y_0^2}\left[\underbrace{\frac{x_0}{\sqrt{x_0^2+y_0^2}}}_{\alpha_0} + i\underbrace{\frac{y_0}{\sqrt{x_0^2+y_0^2}}}_{\beta_0}\right]$$
$$= |z_0|(\alpha_0 + i\beta_0)$$

where $\alpha_0, \beta_0 \in \mathbb{R}$ & $\alpha_0^2 + \beta_0^2 = 1$
therefore $\exists\, \theta \in \mathbb{R}$
s.t $\alpha_0 = \cos\theta$
$\beta_0 = \sin\theta$

now as $e^z = 1 + z + \frac{(z)^2}{2!} + \frac{(z)^3}{3!} + \dots$ ← Taylor series (will come later)

now, $|e^{i\theta}|^2 = e^{i\theta}\,\overline{e^{i\theta}}$

n^{th} partial sum for $e^z = 1 + z + \dots \frac{(z)^{n-1}}{(n-1)!}$

for $z = i\theta$

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots \frac{(i\theta)^{n-1}}{(n-1)!}$$
$$= 1 + i\theta - \frac{\theta^2}{2!} - \frac{i\theta^3}{3!} + \frac{\theta^4}{4!} + \dots \frac{(i\theta)^{n-1}}{(n-1)!}$$

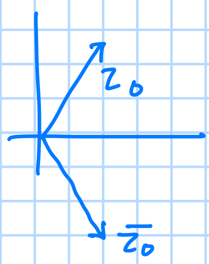
$\overline{e^{i\theta}} = e^{-i\theta}$ using taylor series
so

$$\overline{e^{i\theta}}\, e^{i\theta} = e^{-i\theta + i\theta} = 1 = |e^{i\theta}|^2$$

Trignometric funtions :

for $\theta \in \mathbb{R}$, define $\cos(\theta) = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots$

$\sin(\theta) = \theta - \frac{\theta^3}{3!} \dots$



now, given $z_0 \in \mathbb{C}$ we have

$$z_0 = \left(\frac{z_0 + \bar{z}_0}{2}\right) + i\left(\frac{z_0 - \bar{z}_0}{2i}\right)$$

now $\frac{z_0+\bar{z}_0}{2} \in \mathbb{R}$ as

$$\overline{\left(\frac{z_0+\bar{z}_0}{2}\right)} = \frac{\bar{z}_0+z_0}{2}$$

and $\frac{z_0-\bar{z}_0}{2i} \in \mathbb{R}$ as

$$\overline{\left(\frac{z_0-\bar{z}_0}{2i}\right)} = \frac{\bar{z}_0-z_0}{-2i} = \frac{z_0-\bar{z}_0}{2i}$$

now, $z_0 = |z_0|\left(\left(\frac{z_0+\bar{z}_0}{2|z_0|}\right) + i\left(\frac{z_0-\bar{z}_0}{2i|z_0|}\right)\right)$

For a Right angle triangle, cosine of θ

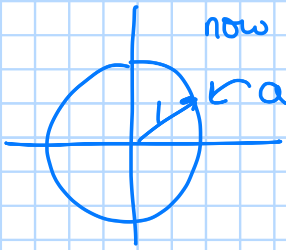
$$\cos\theta = \frac{\text{base}}{\text{hypo}}$$

$$\sin\theta = \frac{\text{perp}}{\text{hypo}}$$

for $0<\theta<\pi$
$Re(z)>0$ and $Im(z)>0$

def: $\cos\theta = \frac{z+\bar{z}}{2|z|}$ $\sin\theta = \frac{z-\bar{z}}{2i|z|}$

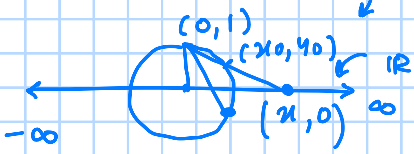
now insted of 0 to π, we have 0 to 2π



← all $|z|=1$
then $z = \cos\theta + i\sin\theta$

Note: In future, we want to see analytic geometry and want to define a notion of "∞" on $\mathbb{C}$.

$|z| \to \infty$ means distance of z from 0 is growing
Topology − 1 point complexification



If we remove $(0,1)$ then there is 1-1 correspondance between $\mathbb{R}$ and points on the line.

line: $y-1 = \frac{y_0-1}{x_0}(x-0)$

for $y=0$

$$-1 = \left(\frac{y_0-1}{x_0}\right)x$$

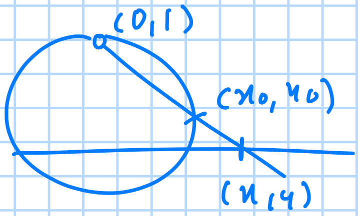
$$x = \frac{x_0}{1-y_0}$$

$\left(\frac{x_0}{1-y_0}, 0\right)$ is a point
where $x_0^2+y_0^2=1$

so if $x = \frac{x_0}{1-y_0}$ and $x_0^2+y_0^2=1$

$x^2 = x_0^2 / 1 - \sqrt{1-x_0^2}$

$x_0 = \frac{2x}{x^2+1}$ and $y_0 = \frac{x^2-1}{x^2+1}$ using $x_0^2 + y_0^2 = 1$



$\left(\frac{y_0+1}{x_0-0}\right) = \left(\frac{0+1}{x-0}\right)$

$x = \frac{x_0}{1-y_0}$ and now,

$x_0^2 + y_0^2 = 1$

$y_0 = \sqrt{1-x_0^2}$

$x - xy_0 = x_0$

$\frac{x - x_0}{x} = \cancel{x} y_0$

$y_0^2 = \left(1 - \frac{x_0}{x}\right)^2 = 1 + \frac{x_0^2}{x^2} - \frac{2x_0}{x}$

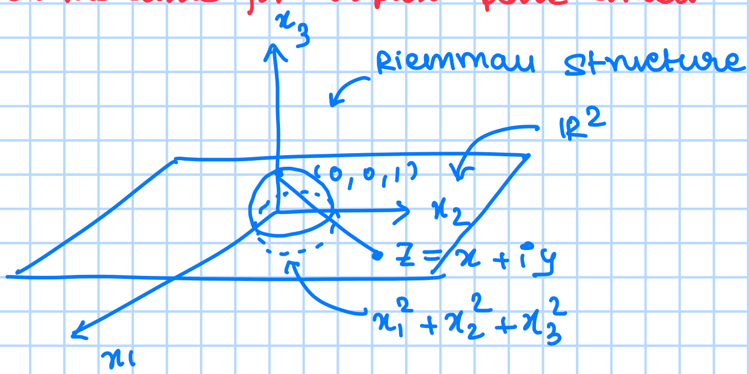
$\cancel{1} - x_0^2 = \cancel{1} + \frac{x_0^2}{x^2} - \frac{2x_0}{x}$

$\frac{2}{x} = \frac{x_0}{x^2} + x_0$

$\frac{\frac{2}{x}}{\frac{1}{x^2} + 1} = x_0$

$x_0 = \frac{2x}{1+x^2}$

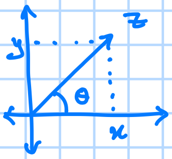
Note: we can do the same for complex plane area:



Here $x = \frac{x_1}{1-x_3}$ $y = \frac{x_2}{1-x_3}$

10th Jan :

Remark :



$y''(x) + y(x) = 0 \rightarrow D^2 + 1 = 0 \Rightarrow D = \pm i$

$\Rightarrow y(t) = c_1 \cos t + c_2 \sin t$

① $y(0) = 0$, $y'(0) = 1$ } defines $\sin(x)$

② $y(0) = 1$, $y''(0) = 0$ defines $\cos(x)$

$(\sin(x))' = \cos(x)$

$(\cos(x))' = -\sin(x)$

$\sin^2 x + \cos^2 x = 1$

Cauchy-Riemann equations:

Reminder : A fn $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be continuous at $x = a$ iff

if $\lim_{x \to a} f(x) = f(a)$

The $\lim_{x \to a} f(x)$ is said to exist if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t

$|f(x) - A| < \varepsilon$ for $|x - a| < \delta$, $x \neq a$

Note: use similar defination for $f : \mathbb{C} \rightarrow \mathbb{C}$ since $|\cdot| ; \mathbb{C} \rightarrow \mathbb{R}_{\geq 0}$

defind by $|z|^2 = z \cdot \bar{z}$, $|z| \geq 0$, for $z \in \mathbb{C}$

observe :

If $f(x)$ is continious at $x = a$, then

1) $\bar{f}(x)$ is also continious at $x = a$
2) $Re(f)(x) = Re(f(x))$ is cont at $x = a$ ← see
3) $Im(f)(x) = Im(f(x))$ is cont at $x = a$ ← see
4) $|f(x)|$ is also cont at $x = a$

proof :

1) since $|\bar{w}| = |w|$ for $w \in \mathbb{C}$

$|f(x) - A| < \varepsilon \Rightarrow |\overline{f(x) - A}| < \varepsilon$

$\Rightarrow |\bar{f}(x) - \bar{A}| < \varepsilon$

2) $|Re(w)| \leq |w|$, so here,

$|f(x) - A| < \varepsilon \Rightarrow |Re(f(x) - A)| < \varepsilon$

$\Rightarrow |Re(f(x)) - Re(A)| < \varepsilon$

3) $Im(f(x))$ is same as (2)

4) $|f(x)|$ is for the triagle inequality, that is :

$||f(x)| - |A|| \leq |f(x) - A| < \varepsilon$

$\Rightarrow ||f(x)| - |A|| < \varepsilon$

$\Rightarrow$ $|f(x)|$ is cont

Def: (Derivative in $\mathbb{C}$) Derivative of $f: \mathbb{C} \to \mathbb{C}$ at $z = a$, denoted by $f'(a)$ and equals

$$\lim_{z \to a} \frac{f(z) - f(a)}{z - a} \text{ if it exist}$$

Note: $\lim_{z \to a} |f(z) - f(a)|$

$$= \lim_{z \to a} \left| \frac{f(z) - f(a)}{(z-a)} \cdot (z - a) \right|$$

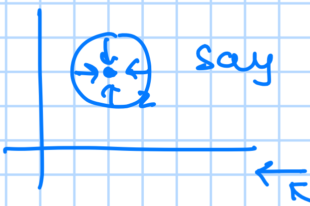
as $|z - a| \to 0$

$$= \lim_{z \to a} \underbrace{\left| \frac{f(z) - f(a)}{z - a} \right|}_{\in \mathbb{R}} |z - a| = 0$$

so Differentiability $\Rightarrow$ continuity

Exe:

for $z \to a$ in all directions



say $f: \mathbb{C} \to \mathbb{C}$ is differentiable at $z = a$

then $f'(a) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \dfrac{f(a+h) - f(a)}{h}$

parallel to real axis

now for imaginary axis:

$$f'(a) = \lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(a + ih) - f(a)}{ih}$$

Note: for Both imaginary and real line, both the values should be equal if f' is diff

since $f: \mathbb{C} \to \mathbb{C}$, write $f(x, y) = u(x, y) + iv(x, y)$ where $z = x + iy$

$$u: \mathbb{R}^2 \to \mathbb{R}$$
$$\& \quad v: \mathbb{R}^2 \to \mathbb{R}$$

Exe: If f is diff at $z = a$, what can we say about partial derivatives of $u(x,y)$ & $v(x,y)$

Here if parallel to x:

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}}} \frac{f(a+h) - f(a)}{h} = f'(a) \text{ then}$$

$$\frac{\partial}{\partial x}(u(x,y) + iv(x,y)) \text{ exist}$$

and similarly for y:

$$\frac{\partial}{i\,\partial y}\Big(u(x,y) + iv(x,y)\Big) \text{ exist}$$

now $\frac{\partial}{\partial x}(u(x,y)+iv(x,y))$ exist $\quad$ ← $f(z)$ exist ⇒ $\overline{f(z)}$ exist
$\mathbb{R}^2 \to \mathbb{R}$

then $\frac{\partial}{\partial x}(u(x,y)-iv(x,y))$ also exist
(so the individual also exist)

$$\lim_{z\to a}\frac{f(a)-f(z)}{a-z} = \lim_{h^2+k^2\to 0}\frac{f(a+(h+ik))-f(a)}{h+ik}$$

so $\frac{\partial}{\partial x}u(x,y)$, $\frac{\partial}{\partial y}u(x,y)$, $\frac{\partial}{\partial x}v(x,y)$, $\frac{\partial}{\partial y}v(x,y)$ all exist at $a=x+iy$

Also,

$$f'(a) = \lim_{\substack{h\to 0\\ h\in\mathbb{R}}}\frac{f(\alpha+i\beta+ih)-f(\alpha+i\beta)}{ih}$$

$$= \frac{\partial}{i\partial y}(u(x,y)+iv(x,y))$$

$$= \lim_{\substack{h\to 0\\ h\in\mathbb{R}}}\frac{f(a+h)-f(a)}{h}$$

$$= \frac{\partial}{\partial x}(u(x,y)+iv(x,y))$$

Note:

$$\frac{\partial}{\partial x}(u(x,y)+iv(x,y)) = \frac{\partial}{i\partial y}(u(x,y)+iv(x,y))$$

$$\frac{\partial}{\partial x}u + i\frac{\partial}{\partial x}v = -i\frac{\partial}{\partial y}u + \frac{\partial}{\partial y}v$$

so

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial}{\partial y}v}$$

and

$$\boxed{\frac{\partial v}{\partial x} = -\frac{\partial}{\partial y}u}$$

These are called Cauchy-Riemann equation

Assume if $f,g:\mathbb{C}\to\mathbb{C}$ is s.t f,g are cont. at $z=a$, then the $f+g$ and $f\cdot g$ is also cont at $z=a$

$$\frac{\partial u}{\partial x}=u_x \quad \frac{\partial u}{\partial y}=u_y \quad \frac{\partial v}{\partial x}=v_x \quad \frac{\partial v}{\partial y}=v_y$$

now, $u_x = v_y$
$u_y = -v_x$

$$|f'(z)|^2 = |u_x + iv_x|^2$$

$$|f'(z)|^2 = u_x^2 + v_x^2 = u_x v_y + v_x(-u_y)$$

$$= \det\begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = \text{Jacobian of } u(x,y) \text{ \& } v(x,y) \text{ wrt } x,y$$

$$|f'(z)|^2 = \det \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = \text{Jacobian}$$

Exe: what if $u(x,y), v(x,y) : \mathbb{R}^2 \to \mathbb{R}$ have continuous partial derivatives & satisfies the C-R equations, then what about $\mathbb{C}$-differentiability of $f(z) = u(x,y) + iv(x,y)$ for $z = x+iy$

more stronger
Extra

$$u(x+h, y+k) - u(x,y) = \frac{\partial u}{\partial x}(x,y)\cdot h + \frac{\partial u}{\partial y}(x,y)\cdot k + \gamma_1(h,k)$$

where $\lim\limits_{h^2+k^2 \to 0} \dfrac{\gamma_1(h,k)}{\sqrt{h^2+k^2}} = 0$

Hint: $u(x+h, y+k) - u(x,y)$

$= u(x+h, y+k) - u(x, y+k) + u(x, y+k) - u(x,y)$

$= \partial/\partial x\,[u(x, y+k)]h + \partial/\partial y\,[u(x,y)]\cdot k$

as $\sqrt{h^2+k^2} \to 0$

now $\left(\partial/\partial x\, u(x, y+k)\right)\cdot k = \left(\partial/\partial x\, u(x,y)\right)k + o(h,k)$

Similarly $v(x+h, y+k) - v(x,y) = \dfrac{\partial v}{\partial x}(x,y)\cdot h + \dfrac{\partial v}{\partial y}(x,y)\cdot k + \gamma_2(h,k)$

s.t $\lim\limits_{h^2+k^2 \to 0} \dfrac{\gamma_2}{\sqrt{h^2+k^2}} = 0$

goal: $\lim\limits_{\sqrt{h^2+k^2} \to 0} \dfrac{f(z+h+ik) - f(z)}{h+ik}$ exist

$f(z) = u(x,y) + iv(x,y)$

$f(z+h+ik) - f(z) = u(x+h, y+k) + iv(x+h, y+k) - u(x,y) - iv(x,y)$

$= u_x h + u_y k + \gamma_1(h,k) + i[v_x h + v_y k + \gamma_2(h,k)]$

$= \underbrace{[u_x + iv_x](h+ik)}_{} + \gamma_1(h,k) + i\gamma_2(h,k)$

$u_x h + u_y k + i v_x h + i v_y k$

$= u_x h - v_x k + i v_x h + i u_x k$

$= (u_x + i v_x)(h+ik)$

$$\lim_{\sqrt{h^2+k^2} \to 0} \frac{f(z+h+ik) - f(z)}{h+ik}$$

$$= \lim_{\sqrt{h^2+k^2}\to 0} u_x + i v_x + \lim_{\sqrt{h^2+k^2}\to 0} \overbrace{r_1(h,k) + i r_2(h,k)}^{\to 0}$$

$$= \lim_{\sqrt{h^2+k^2}\to 0} u_x + i v_x$$

$$= u_x + i v_x$$

Note: so if we have u, v s.t they are cont partial derivative & satisfies C-R then

$$f(z) = u + iv \text{ is } \mathbb{C}\text{-differentiable}$$

later: If $f(z)$ is $\mathbb{C}$-diff then so is $f'(z)$

$\mathbb{C}$-diff $\cong$ Analytic — see this later

laplacian Δ $C_2(\mathbb{R}^2) \to$

14th Jan :

Last time: complex diff equation

$$f(z) = u(x,y) + iv(x,y)$$
$$z = x+iy$$

is $\mathbb{C}$-diff, then
$$u_x = v_y$$
$$\& \quad u_y = -v_x$$

Eg: 1) cost fuction
$$f_0(z_1) = c \in \mathbb{C}$$
$$\frac{\partial}{\partial z} f_0(z) = 0$$
$$\lim_{h\to 0} \frac{f_0(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{c-c}{h} = 0$$
$$h \in \mathbb{C}$$

2) polynomials :

$f_n(z) = z^n$ is $\mathbb{C}$-diff as

$$\lim_{h\to 0} \frac{f_n(z+h) - f_n(z)}{h} = \lim_{h\to 0} \frac{(z+h)^n - z^n}{h}$$
$$= \lim_{h\to 0} \frac{z^n + {}^nC_1 z^{n-1} h + \dots {}^nC_n h^n - z^n}{h}$$
$$= {}^nC_1 z^{n-1} = n z^{n-1}$$

$$\frac{d}{dz} f_n(z) = n z^{n-1}$$

so for $a_n \in \mathbb{C} \Rightarrow a_n z^n$ is also $\mathbb{C}$-diff

$\Rightarrow P_n(z) = a_0 + a_1 z + \dots\dots + a_n z^n$ is $\mathbb{C}$-diff

Rational function:

$f(z) = \frac{P(z)}{q(z)}$ defined by $z \in \Omega \subseteq \mathbb{C}$, a domain inside $\mathbb{C}$ simply connected

say, $q(z) \neq 0$ on Ω, $P(z)$ & $q(z)$ is diff on Ω.

then $$f'(z) = \frac{P'(z)q(z) - P(z)q'(z)}{q^2(z)}$$

$$\frac{d}{dz} f(z) = \lim_{h\to 0} \frac{\frac{P(z+h)}{q(z+h)} - \frac{P(z)}{q(z)}}{h}$$
$$= \lim_{h\to 0} \frac{P(z+h)q(z) - P(z)q(z+h)}{q(z+h)q(z)\, h}$$

$$= \lim_{h \to 0} \frac{[P(z+h) - P(z)]\, q(z) + P(z)\, [q(z) - q(z+h)]}{q(z+h)\, q(z)\, h}$$

$$= \frac{P'(z)\, q(z) - q'(z) P(z)}{(q(z))^2}$$

Note: $\mathbb{R}$ is a complete field

$\rightarrow$ every cauchy sequence converges

claim : $\mathbb{C}$ is also complete

proof : Say $\{z_n\}_{n\geq 1}$ is cauchy sequence then given $\varepsilon > 0$,

$\exists N \in \mathbb{N}$ s.t
$n, m > N \Rightarrow |z_n - z_m| < \varepsilon$

1) $\{z_n\}_{n\geq 1}$ is a cauchy sequence then $\{Re(z_n)\}_{n\geq 1}$ and $\{Im(z_n)\}_{n\geq 1}$ also cauchy seq^n (in $\mathbb{R}$) $\rightarrow$ Done

2) Say $a = \lim_{n\to\infty} Re(z_n)$ and $b = \lim_{n\to\infty} Im(z_n)$, then $\rightarrow$ Done

$$\lim_{n\to\infty} z_n = a + ib$$

Infinite Series :

via limit of partial sums

$s = \sum_{n=1}^{\infty} a_n$ converges ($a_n \in \mathbb{C}$) if $\{s_n\}_{n\geq 1}$ converges

$$s_n = \sum_{k=1}^{n} a_k$$

Say $\{f_n(z)\}_{n\geq 1}$ is a seq of fns $f_n : \mathbb{C} \to \mathbb{C}$

$\sum_{n\geq 1}^{\infty} f_n(z)$ converges at z if

$\{s_n(z)\}_{n\geq 1}$ converges

when $s_n(z) = \sum_{i=1}^{n} f_i(z)$

take ?

more generally, $\Omega \subseteq \mathbb{C}$ and see/find out the nature of convergence when z varies in Ω.

uniform convergence :

given $\{f_n\}_{n\geq 1}$, $f_n : \Omega \to \mathbb{C}$, the seq $\{f_n(z)\}_{n\geq 1}$ is said to converge uniformly on Ω, if given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t

$$n, m > N \Rightarrow |f_n(z) - f_m(z)| < \varepsilon \ \forall z \in \Omega$$

Eg: $f_n(x) = x\left(1+\frac{1}{n}\right)$ for $x \in \mathbb{R}$, & $n \geq 1$

if we fix x

then

for $\lim_{n\to\infty} f_n(x) = x$

given $\varepsilon > 0$, find $N \in \mathbb{N}$ s.t

$|f_n(x) - f(x)| < \varepsilon \quad \forall\ n > N$

$$\Rightarrow \left|x\left(1+\frac{1}{n}\right) - x\right| = \left|\frac{x}{n}\right| < \varepsilon$$

$$\Rightarrow n > \frac{|x|}{\varepsilon}$$

as n is dependent on x

if n is 100

then putting $|x| > 100\,\varepsilon$

this is false, so function is not uniform convg.

but it does converg to $f(x) = x$

Power series:

A series of the form $a_0 + a_1 z + a_2 z^2 + \cdots$ where $a_n \in \mathbb{C}$
$z \in \mathbb{C}$ is called a power series (with centre at $z = 0$)

more genrally, power series looks like :

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n \in \mathbb{C},\ z_0 \in \mathbb{C},\ z \in \mathbb{C}$$

geometric series:

Series $\sum_{n=0}^{\infty} z^n$ converges when $|z| < 1$

$$s_n = \sum_{k=0}^{n-1} z^n = \frac{1 - z^n}{1 - z}$$

when $|z| < 1$ then
$|z|^n \to 0$
as $n \to \infty$

$$\Rightarrow \sum_{n \geq 0} z^n \text{ convrgs to } \frac{1}{1-z} \text{ for } |z| < 1$$

if $|z| \geq 1$ then $\{s_n(z)\}_{n \geq 1}$ is not a cauchy sequence

consider $s_{n+1}(z) - s_n(z) = z^n$ ← interesting way of proving

$|z| \geq 1 \Rightarrow |z^n| \geq 1$

proving $\{s_n(z)\}$ is not cauchy

Note: $\sum_{n=1}^{\infty} \frac{1}{n} z^n$ converges for $|z| < 1$ on $|z| = 1$ the series may or may not converge

fourier analysis (carleson's therm) $\sum_{n \geq 1}^{\infty} \frac{1}{n} z^n$ cong on $|z| = 1$ for set of measure 1

$\sum_{n=1}^{\infty} \frac{1}{n} z^n$ for $|z| < 1$ $\quad \lim_{n\to\infty} \left| \dfrac{\frac{1}{(n+1)} (z)^{n+1}}{\frac{1}{n} (z)^n} \right| = |z| < 1$

also $\limsup |a_n|^{1/n} = 1$

aim: consider $\sum_{n\geq 0} a_n z^n$ and $\exists R \in \mathbb{R}$, $0 \leq R \leq \infty$ series converges abs. when $|z|<R$ divergs when $|z|>R$.

proof: define

$$R = \begin{cases} \dfrac{1}{\limsup |a_n|^{1/n}} & \text{if } 0<\limsup|a_n|^{1/n}<\infty \\ 0 & \text{if } \limsup |a_n|^{1/n} = \infty \\ \infty & \text{if } \limsup |a_n|^{1/n} = 0 \end{cases}$$

$R=0$ then series only conv for $z=0$

1) If $|z|<R$ then $\sum_{n\geq 0} a_n z^n$ converges absolutely

take $\rho \in \mathbb{R}$ s.t $|z|<\rho<R$

by def of limsup & $\rho<R$

$\Rightarrow \frac{1}{\rho} > \frac{1}{R}$

$\Rightarrow \frac{1}{\rho} > \limsup |a_n|^{1/n}$

$\Rightarrow \frac{|z|}{\rho} > \limsup |z||a_n|^{1/n}$

By def of limsup

$\Rightarrow \exists N$ s.t $n>N \Rightarrow \left(\frac{|z|}{\rho}\right)^n > |z|^n a_n$

this is <1

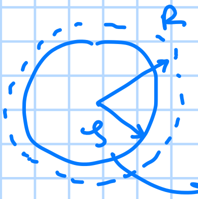
when $|z|<R$

$$s_\infty = \sum_{n=0}^{\infty} |a_n z^n| = \sum_{n=0}^{N} |a_n z^n| + \underbrace{\sum_{n=N+1}^{\infty} |a_n z^n|}_{}$$

$|a_n z^n| < \left(\frac{|z|}{\rho}\right)^n < 1$

$$\leq \underbrace{\sum_{n=0}^{N} |a_n z^n|}_{<\infty} + \underbrace{\sum_{n=N+1}^{\infty} \left|\frac{z}{\rho}\right|^n}_{<\infty}$$

for $|z| \leq \rho < R$, $\sum a_n z^n$ converges uniformly



uniform conv

$|z| \leq \rho$

choose ρ' s.t

$\rho < \rho' < R$

choose ρ' s.t $\rho<\rho'<R$

cauchy sequence $\{s_n(z)\}$ where

$s_n(z) = \sum_{k=0}^{n} a_k z^n$ converges uniformly

find $N \in \mathbb{N}$ s.t $n, m > N \Rightarrow |s_n(z) - s_m(z)| < \varepsilon$ $\forall |z| \leq \rho$

Say $n>m$, $|s_n(z)-s_m(z)|$

$$=\left|\sum_{k=m+1}^{n} a_k z^k\right|$$

Here we can prove that this term goes to 0 or we can use M-test $\rightarrow$ $\leq \sum_{k=m+1}^{n} |a_k z^k|$

we defn of $\limsup |a_n|^{1/n}$ $\rightarrow$ By defn, $\exists N$ s.t $k>N$

$$\Rightarrow |a_k z^k| < \left(\frac{|z|}{\rho}\right)^k \; \& \; \left(\frac{|z|}{\rho'}\right)^k$$

M-test:
$\sum_{1}^{\infty} f_i(z)$ will conv on X if $\exists$ conv seq s.t $\sum M_n$ conv
$|f_i(z)| \leq M_i \;\; \forall i,\; z\in X$

$$\left(\frac{|z|^k}{\rho'}\right) < \left(\frac{\rho}{\rho'}\right)^k \Rightarrow \sum_{n=1}^{N} |a_n z^n| + \underbrace{\sum_{n=N+1}^{\infty} |a_n z^n|}$$

each term $\leq (\rho/\rho')^n$ — this is M_n — use weierstrass M test

as $M_n = (\rho/\rho')^n$ ← this converges as $\rho/\rho' < 1$

2) If $|z|>R$, the series diverges, By defn of lim sup there are ∞-many n's s.t

$$|a_n z^n| > 1 \Rightarrow |a_n z^n| \not\to 0$$

$$\Rightarrow \text{series diverges}$$

Claim: $\mathbb{C}$ is also complete

proof: Say $\{z_n\}_{n\geq 1}$ is cauchy sequence then given $\varepsilon>0$,

$\exists N\in\mathbb{N}$ s.t
$n,m>N \Rightarrow |z_n - z_m|<\varepsilon$

1) $\{z_n\}_{n\geq 1}$ is a cauchy sequence then $\{Re(z_n)\}_{n\geq 1}$ and $\{Im(z_n)\}_{n\geq 1}$ also cauchy seqn (in $\mathbb{R}$)

as $\forall \varepsilon>0$, $\exists N\in\mathbb{N}$ s.t $\forall n,m\geq N$

$$|z_n - z_m|<\varepsilon$$
$$\Rightarrow \left((Re(z_n)-Re(z_m))^2 + (Im(z_m)-Im(z_n))^2\right)^{\frac{1}{2}} < \varepsilon$$
$$\Rightarrow (Re(z_n)-Re(z_m))^{2\cdot 1/2} < \varepsilon$$
$$\Rightarrow |Re(z_n)-Re(z_m)|<\varepsilon$$
$$\Rightarrow \{Re(z_n)\}_{n\geq 1} \text{ is cauchy in } \mathbb{R}$$

sim $\{Im(z_n)\}_{n\geq 1}$ is cauchy in $\mathbb{R}$

2) Say $a=\lim_{n\to\infty} Re(z_n)$ and $b=\lim_{n\to\infty} Im(z_n)$, then

as z_n is cauchy
$\forall \varepsilon>0$ $\exists N_0\in\mathbb{N}$ s.t $\forall n\geq n_0$
$|Re(z_n)-a|<\varepsilon$

$\forall \varepsilon > 0, \exists N_1 \in \mathbb{N}$ s.t $\forall n \geq N_1$,

$|Im(z_n) - b| < \varepsilon$

now $|Re(z_n) + i\,Im(z_n) - a - ib|$

$\leq |Re(z_n) - a| + |Im(z_n) - b|$

as $Re(z_n) - a = z_1$
$i\,Im(z_n) - ib = z_2$
then
$|z_1 + z_2| \leq |z_1| + |z_2|$

$\Rightarrow |Re(z_n) + i\,Im(z_n) - a - ib| < 2\varepsilon$

$\forall n \geq \max\{N_0, N_1\}$

$\therefore \forall \varepsilon' = 2\varepsilon > 0, \exists N = \max\{N_0, N_1\}$ s.t

$|z_n - a - ib| < \varepsilon' = 2\varepsilon$
$\forall n \geq N$

$\Rightarrow z_n \longrightarrow a + ib$

$\therefore$ every seq in $\mathbb{C}$ converges

17th Jan :

Power series:

$$\sum_{n\geq 0} a_n z^n \qquad R^{-1} = \limsup |a_n|^{1/n}$$

Idea. $\sum_{n<N} a_n z^n + \underbrace{\sum_{n\geq N} a_n z^n}_{|a_n z^n| < (\rho/R)^n < (1)^n}$

$\overline{\lim} |a_n|^{1/n} \leq \overline{\lim} \left|\frac{a_{n+1}}{a_n}\right|$

Real analysis: If $\{a_n\}$ is a sequence of non-zero real numbers, then

$$\liminf\left|\frac{a_{n+1}}{a_n}\right| \leq \liminf |a_n|^{1/n} \leq \limsup_{n\to\infty} |a_n|^{1/n} \leq \limsup_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$ ← from real analysis

Claim: $f(z) = \sum_{n\geq 0} a_n z^n$ defined for $|z|<R$ (when $R>0$) is $\mathbb{C}$-differentiable in $|z|<R$,

$$f'(z) = \sum n a_n z^{n-1} \text{ for } |z|<R$$

proof: consider

$$f_1(z) = \sum_{n\geq 1} n a_n z^{n-1} \quad (\text{goal } f_1(z) = f'(z) \text{ for } |z|<R)$$

to show that radius of convergence for $f_1(z)$ is also R.

Say R_1 = Radius of conv of $f_1(z)$ then

for R_1' = Radius of convergence for $z f_1(z) = \sum n a_n z^n$

$$(R_1')^{-1} = \limsup |n a_n|^{1/n}$$

(as product of two seq whose lim sup exist is first comb of both)

$$= \limsup |n|^{1/n} . \limsup |a_n|^{1/n}$$

$$= 1 \times \limsup |a_n|^{1/n}$$

$$= \limsup |a_n|^{1/n} = (R)^{-1}$$

$|n|^{1/n} = 1+\delta_n$ for $\delta_n \geq 0$

$|n| = (1 + {}^nC_1 \delta_n + {}^nC_2 \delta_n^2 + \dots {}^nC_n \delta_n^n$

$> {}^nC_2 \delta_n^2$

$|n| > {}^nC_2 \delta_n^2 \Rightarrow \delta_n^2 < \frac{2}{n-1}$

as $n \to \infty$ $\delta_n \to 0$

As radius of $z f_1(z)$ is $R \Rightarrow$ Radius of $f_1(z)$ is R

for $|z| \neq 0$,

$$f_1(z) = \sum_{n\geq 0} (n+1) a_{n+1} (z)^n$$

$$z f_1(z) = \sum_{n\geq 1} (n) a_n z^n$$

$$= z\left[\sum_{n\geq 1} n a_n z^{n-1}\right]$$

If $g(z)$ defines a function for $|z|<R_1$, $z\neq 0$ then $\frac{1}{z} g(z)$ also does.

($z f_1(z)$ conv for $z \Leftrightarrow f_1(z)$ conv for z, $z\neq 0$

$z f_1(z)$ conv for z_0 then $z_0 f_1(z_0) = \alpha$ → epsilon delta can also be used

$\Leftrightarrow f_1(z_0) = \frac{\alpha}{z_0}$)

$f'(z) = f_1(z)$

say z_0 s.t $|z_0| < R$ ← finite polynomial

$$f(z) = \sum_{n \geq 0} a_n z^n = \underbrace{\sum_{n=0}^{N-1} a_n z^n}_{S_N(z)} + \underbrace{\sum_{n=N}^{\infty} a_n z^n}_{R_N(z)}$$

$$\left| \frac{S_N(z) - S_N(z_0)}{z - z_0} - S_N'(z_0) \right|$$

To show

$$\frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0) \to 0$$

as $z \to z_0$

$$= \frac{S_N(z) + R_N(z) - S_N(z_0) - R_N(z_0)}{z - z_0} - f_1(z_0)$$

subtracted

$$= \left[\frac{S_N(z) - S_N(z_0)}{z - z_0} - S_N'(z_0) \right]$$

$$+ \left[S_N'(z_0) - f_1(z_0) \right] + \left[\frac{R_N(z) - R_N(z_0)}{z - z_0} \right]$$

added

$S_N(z)$ is a poly ⇒ C-diff

given $\varepsilon > 0$, $\exists\, \delta_N > 0$ s.t

① $\left| \frac{S_N(z) - S_N(z_0)}{z - z_0} - S_N'(z_0) \right| < \varepsilon/3$ } as $S_N(z)$ is diff this exist $\forall N$

goal: given $\varepsilon > 0$, find $\delta > 0$ s.t $0 < |z - z_0| < \delta$

$$\Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - f_1(z) \right| < \varepsilon$$

② $\frac{R_N(z) - R_N(z_0)}{z - z_0} = \sum_{n=N}^{\infty} a_n \frac{(z^n - z_0^n)}{z - z_0}$ ← Rearranged as abs conv (proved as abs conv)

Binomial

$$= \sum_{n=N}^{\infty} a_n \left(z^{n-1} + z^{n-2} z_0 + \dots + z_0^{n-1} \right)$$

$$\Rightarrow \left| \frac{R_N(z) - R_N(z_0)}{z - z_0} \right| \leq \sum_{n=N} |a_n| \sum_{a=0}^{n-1} |z|^a |z_0|^{n-1-a}$$

$|z| < R$ choose $\rho < R$ s.t

$|z|, |z_0| < \rho < R$

$$\leq \underbrace{\sum_{n=N}^{\infty} |a_n|\, n \rho^{n-1}}_{\text{Reminder of conv power series } f_1(z)}$$

choose N big s.t

$$\left| \frac{R_N(z) - R_N(z_0)}{z - z_0} \right| < \varepsilon/3$$

By defn of $f_1(z)$, $\lim_{n\to\infty} S_N'(z_0) = f_1(z_0)$

choose N_1 s.t

$$|S_{N_1}'(z_0) - f_1(z_0)| < \varepsilon/3$$

Choose $M > N, N_1$ $\Rightarrow$ choose δ

s.t $|z - z_0| < \delta$, $|z|, |z_0| < R$

$$\Rightarrow \left| \frac{S_n(z) - S_n(z_0)}{z - z_0} - S_n'(z_0) \right| < \varepsilon/3 \text{ for } n = M$$

$$\Rightarrow \left| \frac{f(z) - f(z_0)}{z - z_0} - f_1(z_0) \right| < \varepsilon$$

$\Rightarrow f'(z) = f_1(z)$ for $|z| < R$

Defn: exponential function

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} \quad \text{where } 0! = 1$$

see R ($R = \infty$, but check)

also $\exp(z) = e^z$

Real powers: For $a, b > 0$, say $b \in \mathbb{N}$, $a^b = \underbrace{a \times a \times \cdots a}_{b \text{ times}}$

$a^{1/b}$ is positive real s.t

$$\underbrace{(a^{1/b}) \cdot \quad \cdot \quad (a^{1/b})}_{b \text{ times}} = a$$

$\Rightarrow a^{p/q}$ for $a > 0$, $p/q > 0$ makes sense

as $\mathbb{Q}_{\geq 0}$ is dense in $\mathbb{R}_{\geq 0}$

a^b for $b \in \mathbb{R}_{\geq 0}$ is

$$\lim_{p/q \to b} a^{p/q} = a^b$$

and $a^{-b} = 1/(a^b)$ by additivity of powers

So we end up defing $a^b \ \forall b \in \mathbb{R}, a \in \mathbb{R}_{>0}$

$\Rightarrow e^t$ is defined for $t \in \mathbb{R}$

&

$$e = \sum_{n \geq 0} \frac{1}{n!}$$

21st Jan :

power series $f(z) = \sum_{n\geq 0} a_n z^n$ has radius $R>0$

$$f_1(z) = \sum_{n\geq 1} n a_n z^{n-1}$$

R. of con for $f_1(z)$ = R. of con $f(z)$

Shown last time : $f'(z) = f_1(z)$ when $|z|<R$

Say Radii of convergence of $f_1(z)$ is R_{f_1}

$$g(z) = z f_1(z) = \sum_{n\geq 1} n a_n z^n$$

now, $\underbrace{\limsup |n a_n|^{1/n} = \limsup |a_n|^{1/n}}_{\text{to show this}}$

first if $z=0$, $\underset{=a_1}{f_1(0)} \leftarrow$ converges (trivial)

take : $z \neq 0$: last time:

since $\lim_{n\to\infty} |n|^{1/n} = 1$

radius of g say $R_g = R$

say $|z|<R$, $z\neq 0$

$f(z)$ converges means for $\varepsilon>0$, $\exists N$

$m>n>N$

$$\left|\sum_{k=n+1}^{m} a_k z^k\right| < \varepsilon$$

$\forall\ |z|<R$

now as $g = z f_1(z)$

has $R = R_g$

we have $\exists M>0$ $m>n>M$

$\Rightarrow \left|\underbrace{\sum k a_k z^k}_{g}\right| < \varepsilon$

$\Rightarrow$

$|z| \left|\sum k a_k z^{k-1}\right| < \varepsilon$

$\Rightarrow \left|\underbrace{\sum k a_k z^{k-1}}_{f_1(z)}\right| < \varepsilon/|z|$

$\Rightarrow$ $f_1(z)$ converges

$\Rightarrow$ $R_{f_1} \geq R$

if $|z|>R \Rightarrow g(z)$ doesn't converge as $R_g = R$

and $|z| \neq 0$

$\underbrace{\frac{1}{z} g(z)}_{f_1(z)}$ also does not converge

$R_{f_1} \leq R \Rightarrow R_{f_1} = R$

Power series:

$\mathrm{Exp}(z) = \sum_{n\geq 0} \frac{1}{n!} z^n$ where $0! = 1$

Exe: show that $R = \infty$

(Hint: $\overline{\lim}\left(\frac{1}{n!}\right)^{1/n} = 0$

by using $\underline{\lim}\left|\frac{a_{n+1}}{a_n}\right| \leq \underline{\lim}\,|a_n|^{1/n} \leq \ldots$)

$\left(\left|\frac{a_{n+1}}{a_n}\right| = \frac{\frac{1}{(n+1)!}}{\frac{1}{(n)!}} = \frac{1}{(n+1)} \to 0$ as $n\to\infty$ so $\overline{\lim}\,|a_n|^{1/n} \to 0$ by sandwich $R\to\infty\right)$

Notion: $\exp(z) = e^z$ (this is just definational use)

Real analysis: $e = \sum_{n\geq 0} \frac{1}{n!}$, Define $e^t = \lim_{p/q\to t} e^{p/q}$

Exe: why is it that $e^t = \sum_{n\geq 0} \frac{t^n}{n!}$ for $t\in\mathbb{R}$

If we use taylor expansion, we have to prove

$\lim_{h\to 0} \frac{e^h - 1}{h} = 1 \to$ not possible

approach-1: If we can show $e^{a+b} = e^a . e^b$ $\forall\, a,b\in\mathbb{R}$ then

for $p\in\mathbb{Z}$, $e^p = e\times\cdots\times e$

$e^2 = \left(\sum \frac{1}{n!}\right)\left(\sum\frac{1}{m!}\right) \to e^p = \sum\frac{p^n}{n!}$ $p\in\mathbb{Z}$

$e^{1/q} = \sum \frac{(1/q)^n}{n!}$ for $q\in\mathbb{Z}$

$e^{p/q} = (e^p)^{1/q}$

$e^t = \lim_{p/q\to t} e^{p/q}$

Approach 2:

prove $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$

as $\left(1+\frac{1}{n}\right)^n = 1 + \frac{{}^nC_1}{n} + \frac{{}^nC_2}{n^2} + \cdots \frac{{}^nC_n}{n^n}$

${}^nC_1/n = 1$ as $n\to\infty$

${}^nC_2/n^2 \to 1/2$ as $n\to\infty$

so $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \to 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \cdots$

$\Rightarrow e^t = \left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right]^t$

$\Rightarrow e^t = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{nt}$ $\quad nt = r$

$e^t = \lim_{r\to\infty}\left(1+\frac{t}{r}\right)^r = \lim_{r\to\infty}\left(1 + \frac{{}^rC_1}{(r)} t + \frac{{}^rC_2}{(r)^2} t^2 + \cdots\right)$

$= \left(1 + \frac{t}{1} + \frac{t^2}{2!} + \cdots\right) = \sum_{n=0}^{\infty} \frac{t^n}{n!}$

$\exp(z) = \sum_{n \geq 0} \frac{z^n}{n!}$ (def)

$\frac{d}{dz} \exp(z) = \sum_{n \geq 1} \frac{n z^{n-1}}{n!}$ (this is from our discussion before)

$\frac{d}{dz} \exp(z) = \sum_{m \geq 0} \frac{z^m}{m!} = \exp(z)$

so e^z is its own derivative if we define it using power series

Another defination of e^t for $t \in \mathbb{R}$: e^t is the unique soln of $f'(t) = f(t)$ and $f(0) = 1$

($\log|f| = t + c$, $f(t) = e^c t$, $f(0) = 1 \Rightarrow c = 1$)

Claim: If $a, b \in \mathbb{C}$ then $\exp(a+b) = \exp(a) \cdot \exp(b)$

proof: Consider $g(z) = \exp(z) \cdot \exp(a-z)$ where a is fixed, z is variable

$$\partial/\partial z\, g(z) = \left(\partial/\partial z \exp(z)\right) \exp(a-z) - \left(\partial/\partial z \exp(a-z)\right)\left(\exp(z)\right)$$

$\to$ chain rule

$= \exp(z)\exp(a-z) - \exp(a-z)\exp(z)$
$= 0$

$\Rightarrow g'(z) = 0 \quad \forall z$

$\Rightarrow g(z)$ is constant (Exe in problem set-2)

Say $g(z) = c$ for some $c \in \mathbb{C}$

$\Rightarrow \exp(z) \cdot \exp(a-z) = c$

$\Rightarrow$ put $z = 0$

$\exp(0) \cdot \exp(a)$

$$= \underbrace{\left(\sum \frac{0}{n!}\right)}_{=1}\left(\sum \frac{a}{n!}\right) = c$$

$\Rightarrow \exp(a) = c$

$\Rightarrow \exp(z) \cdot \exp(a-z) = \exp(a)$

$a = a' + b'$
$z = a'$
$b' = a - z$
To get

$$(e^{a'}) \cdot (e^{b'}) = e^{a'+b'}$$

Def: $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$

$\sin(z) = \frac{e^{iz} - e^{-iz}}{2i}$

where $e^z = \sum_{n \geq 0} \frac{z^n}{n!}$

(This is by def $\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$)

$$\Rightarrow \frac{e^{iz}+e^{-iz}}{2} = 1-\frac{z^2}{2!}+\frac{z^4}{4!}-\dots$$

$$= \sum_{n\geq 0} \frac{(-1)^n z^{2n}}{(2n)!} = \cos(z)$$

(Putting $e^{iz} = 1+iz-\frac{z^2}{2!} - \frac{iz^3}{3!}\dots$

$e^{-iz} = 1-iz-\frac{z^2}{2!}+\frac{iz^3}{3!}\dots$

$\frac{e^{iz}+e^{-iz}}{2} = \frac{2}{2}-\frac{2}{2}\frac{z^2}{2!}+\frac{2z^4}{2\,4!}\dots$

$= 1-\frac{z^2}{2!}+\frac{z^4}{4!}+\dots$)

$$\sin(z) = \frac{e^{iz}-e^{-iz}}{2i} = \sum \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

($\sin(z) = z-\frac{z^3}{3!}+\frac{z^5}{5!}\dots$)

Note: we know how derivatives work

$$(\sin(z))' = \left(\frac{e^{iz}-e^{-iz}}{2i}\right)'$$

$$= \frac{(e^{iz})'-(e^{-iz})'}{2i}$$

$$= \frac{e^{iz}(\cancel{i})+e^{-iz}(\cancel{i})}{2\cancel{i}}$$

$$= \frac{e^{iz}+e^{-iz}}{2} = \cos(z)$$

similarly

$$(\cos(z))' = -\sin(z)$$

Note: $(\sin(z))'' = -\sin(z)$

$(\cos(z))'' = -\cos(z)$

this means that $f''(t)+f(t)=0$ if $t=z\in\mathbb{R}$

we get

$e^{it} = \cos(t)+i\sin(t)$

($f''(t)+f(t)=0$ $\forall t\in\mathbb{R}$

then $f(t)=\cos t+i\sin t$ or e^{it}

as $e^{it}=\cos t+i\sin t$)

Note: eulers formula $e^{it} = \cos(t)+i\sin(t)$ $t\in\mathbb{R}$

Note: Demoivres formula:

$$\cos(nt)+i\sin(nt) = e^{nit} = (e^{it})^n = (\cos t+i\sin t)^n$$

Also $\cos(a+b)$ & $\sin(a+b)$ formulas can be used by

$$(\cos(a)+i\sin(b))(\cos(b)+i\sin(b)) = \cos(a+b)+i\sin(a+b)$$

($e^{i(a+b)} = e^{ia}\cdot e^{ib}$

$\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$)

Note:

e^z is periodic

Say $z=a+ib$, $a,b\in\mathbb{R}$

$e^z = e^{a+ib} = e^a[e^{ib}]$

in real analysis e^a is always inc

we want to show that

$$e^{z+2\pi i} = e^{z}$$

Restrict $a=0$, want $e^{ib} = e^{ib+i\omega}$ for some $\omega > 0$

$\Rightarrow e^{0} = e^{i\omega}$ for some $\omega > 0$

goal: prove that $\exists \omega$ s.t $e^{i\omega} = 1$

proof: $\sin(0) = 0$

and $(\sin(x))' = \cos(x)$ for $x \in \mathbb{R}$ > 0 for x not too big)

$$\cos(x) = 1 - \underbrace{\frac{x^2}{2!}}_{0} + \underbrace{\frac{x^4}{4!}}_{0} \dots > 0$$

for $x > 0$ not too big

so $\sin'(x) > 0$

$\therefore \sin(x)$ is inc

on the other hand

$$(\sin(x))' = \cos(x) \leq 1 \text{ as } \overbrace{\sin^2(x) + \cos^2(x) = 1}^{\text{from def of } \sin x \text{ and } \cos x}$$

By FTC

$$\sin(x) = \int_0^x (\sin(y))' \, dy = \int_0^x \underbrace{\cos(y)}_{\leq 1} dy$$

$$\leq \int_0^x 1 . dy = x$$

$\Rightarrow \sin(x) \leq x$ ——— ①

Similarly there is a lower bound on cosine

as: $$\cos(x) - 1 = \int_0^x \cos(y)' dy$$

$$= \int_0^x -\sin(y) dy$$

$$\geq \int_0^x (-y) dy$$

$$\cos x - 1 \geq -x^2/2$$

$\Rightarrow \cos x \geq 1 - x^2/2$ ——— ②

(now we have $\sin x \leq x$ and $\cos x \geq 1 - x^2/2$ this is true for all x)

24^{th} Jan :

$e^z = e^a . e^{ib}$ where $z = a+ib$, we wanted to show e^{ib} is periodic

goal: prove that $\exists w$ s.t $e^{iw}=1$

proof : e^{it} is periodic as $t \in \mathbb{R}$
$e^{it} = \cos(t) + i\sin(t)$
$\cos(0)=1 \quad \sin(0)=0$

if we can find $w>0$ s.t $e^{iw}=1$ then e^{it} is periodic
as
$\cos(0)=1$
we have to fin θ s.t
$\cos(\theta)=0$
$\Rightarrow \sin(\theta)=\pm 1 \quad (\cos^2\theta + \sin^2\theta = 1)$
$\Rightarrow e^{i\theta} = \pm i$
$\Rightarrow e^{4i\theta}=1$

as $\sin(x) \leq x \quad \forall x \geq 0$
$(\sin(y))' = \cos(y) \leq 1$

$$\int_0^x (\sin y)' dy \leq \int_0^x (1) dy = x$$

$$\Rightarrow \sin(x) \leq x \quad \text{—— ①}$$

now as $\sin(y) \leq y$

$$\Rightarrow \int_0^x \sin(y) \leq \int_0^x y\, dy$$

$$\Rightarrow -\cos(x) + \cos(0) \leq x^2/2$$

$$\Rightarrow \cos(x) \geq 1 - x^2/2 \quad \text{for } x \geq 0 \quad \text{—— ②}$$

$\cos(y) \geq 1 - y^2/2$

$$\Rightarrow \int_0^x \cos(y)\, dy \geq \int_0^x \left(1 - \frac{y^2}{2}\right) dy$$

$$\Rightarrow \sin(x) \geq x - x^3/6 \quad \text{—— ③}$$
for $x \geq 0$

$\sin(y) \geq y - y^3/6$

$$\Rightarrow \int_0^x \sin(y)\, dy \geq \int_0^x \left(y - \frac{y^3}{6}\right) dy$$

$$\Rightarrow -\cos(x) + 1 \geq \frac{x^2}{2!} - \frac{x^4}{4!}$$

$$\Rightarrow \cos(x) \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \quad \text{—— ④}$$

so we have this: ① $1 - \frac{x^2}{2!} \leq \cos(x) \leq 1 - \frac{x^2}{2!} + \frac{x^4}{4!}$
② $x - x^3/3! \leq \sin(x) \leq x$
} for $x \geq 0$

Note that for $x = \sqrt{3}$

$$\cos(\sqrt{3}) \leq 1 - \frac{3}{2} + \frac{\cancel{9}3}{2\times\cancel{3}\times 4} = -\frac{1}{2} + \frac{3}{8} = -\frac{4}{8} + \frac{3}{8} = -\frac{1}{8} < 0$$

so $\cos(\sqrt{3}) < 0$
$\cos(0) = 1$
so, $\exists\ \theta \in (0, \sqrt{3})$ s.t $\cos(\theta) = 0$

Note : $e^{it + 4\theta n i} = e^{it}$ $\forall\, n \in \mathbb{Z}$
so e^{it} is periodic.

Claim : $\theta \in (0, \sqrt{3})$ s.t $e^{4i\theta} = 1$, then this θ is the smallest such.

proof : let's say there is some ω s.t we have $\omega < \theta$ where $e^{4i\omega} = 1$

now if $\omega < \theta \Rightarrow \omega < \sqrt{3}$

now as $0 < \omega < \sqrt{3}$

$\sin(x) \geq x - x^3/3! > x/2$ when $x \in (0, \sqrt{3})$

(as $x < \sqrt{3}$
$\Rightarrow x^2 < 3$
$\Rightarrow x^3 < 3x$
$\Rightarrow \frac{x^3}{3!} < \frac{x}{2}$
$\Rightarrow x - x^3/3! > x - x/2$)

so $x \in (0, \sqrt{3}) \Rightarrow \sin(x) > 0$
$\Rightarrow (-\cos(x))' > 0$
$\Rightarrow \cos(x) < 0$ for $x \in (0, \sqrt{3})$
$\Rightarrow \cos(x)$ is strictly dec for $(0, \sqrt{3})$
$\Rightarrow \cos(0) = 1 > \cos(\omega) > \cos(\theta) = 0$
$\Rightarrow 1 > \cos(\omega) > 0$
and
$\sin^2(\omega) + \cos^2(\omega) = 1$
$\Rightarrow 0 < \sin(\omega) < 1$
$\Rightarrow e^{i\omega} \neq \pm i, \pm 1$
$\Rightarrow e^{4i\omega} \neq 1$

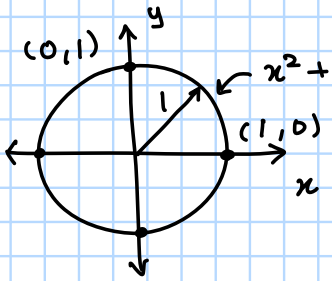
($\omega^4 = 1 \Leftrightarrow \omega = \pm 1, \pm i$
cont on $\omega^n = 1$
$\Leftrightarrow \omega = n$ roots)

Note : θ is the "unique" period i.e any other θ, satisfying $e^{it + 4\theta_1 i} = e^{it}$
satisfying $\theta_1 = n\theta$ for some $n \in \mathbb{Z}$
say $m \in \mathbb{Z}$ s.t
$m\theta < \theta_1 < (m+1)\theta$
$\Rightarrow 0 < \theta_1 - m\theta < \theta$
where $\theta_1 - m\theta$ is also a period
$\Rightarrow \theta_1 - m\theta = \theta$
as θ is lowest in $(0, \sqrt{3})$

($\omega^n = 1$
$= e^{4i\theta}$ for $\theta \in (0, \sqrt{3})$
$\omega^n = e^{4i\theta}$
$\omega = e^{(4i\theta)\frac{1}{n}}$)

Claim: $\pi = 2\theta$ by definition

proof:



$\cos^2(t)+\sin^2(t)=1$
$\cos(0)=1$
$\sin(0)=0$

parametrise circle by $(x(t), y(t)) = (\cos(t), \sin(t))$

length of curve, $\gamma(t) = (x(t), y(t))$ should be

$$\int_0^{4\theta} \text{speed}(\gamma(t))\,dt$$ smallest period

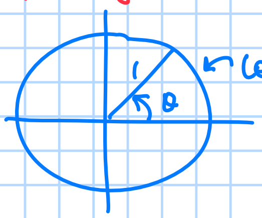
$$= \int_0^{4\theta} \left[\sqrt{\left(\frac{d}{dt}x\right)^2 + \left(\frac{d}{dt}y\right)^2}\right] dt$$

$$= \int_0^{4\theta} \left[\sqrt{\cos^2 t + \sin^2 t}\right] dt$$

$$= \int_0^{4\theta} (1)\,dt = 4\theta$$

now $4\theta =$ circumference of circle
$= 2\pi(r)$
$4\theta = 2\pi$
$\Rightarrow 2\theta = \pi$

Def: (Angle) length of a curve on unit radius when we go by radians



Def: $\log(z)$ for $z \in \mathbb{C}$ is the inverse function of $\exp(z)$

One should talk about the set of values of $\log(z)$ as $\exp(z)$ is a periodic function.

$z = |z| \cdot \frac{z}{|z|}$ $\quad (|z| \neq 0)$

$= |z|(\cos\theta + i\sin\theta)$

for some $\theta \in \mathbb{R}$
and $0 \leq \theta < 2\pi$ smallest such period

So we can uniquely write

$$z = |z|(\cos\theta + i\sin\theta)$$

$|\exp(z)| = |\exp(a+ib)| = |\exp(a)|.1 > 0$

so exp(z) never vanishes

$$\exp : \mathbb{C} \to \underbrace{\mathbb{C}^{\times}}$$

$\mathbb{C}^{\times}$ = set of all complex numbers which has a multiplicative inverse ($\mathbb{C} \setminus \{0\}$)

$$\Rightarrow \log : \mathbb{C}^{\times} \to \mathbb{C}$$

then for $w = e^{z}$, by defination

$$\log(\omega) = z$$

where

$$\omega = |\omega| e^{it} \text{ for some } t$$

$$\Rightarrow \log(\omega) = z = \log|\omega| + it$$

as $z = a + it$

$$e^{z} = e^{a+it} = |\omega| e^{it} = \omega$$

$$|\omega| = e^{a}$$

Note: if $w \in \mathbb{R}_{>0}$ then $\log(w)$ is <u>uniquely</u> <u>defined</u>

Note: In general $\log(\omega)$ has ∞-many values

Eg: as $-1 = e^{-i\pi} = e^{-i\pi + 2\pi i n}$ $\forall n \in \mathbb{Z}$

$$e^{-i\pi + 2\pi i n}$$

$$\Rightarrow \log(-1) = \{-i\pi + 2n\pi i\}_{n \in \mathbb{Z}}$$

$$\log(i) = \log\left[e^{i\pi/2 + 2\pi n i}\right] \quad \forall n \in \mathbb{Z}$$

$$= \left\{ i\frac{\pi}{2} + 2\pi n i \right\}_{n \in \mathbb{Z}}$$

principle value of log:

exactly when $t \in (-\pi, \pi]$

Def: a^{b} for $a, b \in \mathbb{C}$, then $a^{b} := \exp(b \log(a))$ $(a > 0)$

so if $a \in \mathbb{R}_{>0}$ then $a^{b} := \underbrace{\exp(b \underbrace{\log(a)}_{\text{unique}})}_{\text{unique}}$

also if $b \in \mathbb{N}$, $a \in \mathbb{C}$ then

$$\log(a) = \log(|a| e^{i\theta})$$

$$= \log(|a|) + i\theta + 2n\pi i$$

now $a^{b} = \exp(b \log(|a|) + b i\theta + b 2n\pi i)$

$= \exp(b\log(|a|) + bi\theta)$

$= \underbrace{e^{b\log|a| + bi\theta}}_{\text{this is unique as } bn \in \mathbb{Z}}$

Exe: what is the value of i^i?

as $i^i := \exp(i\,\mathrm{Log}(i))$

$= \left\{\exp\left(i\left(i\frac{\pi}{2} + 2i\pi n\right)\right)\right\}_{n\in\mathbb{Z}}$

$= \left\{\exp\left(-\frac{\pi}{2} - 2\pi n\right)\right\}_{n\in\mathbb{Z}}$

Pricipe value of $i^i = \exp(i\underbrace{\log(i)}_{(-\pi,\pi]})$

$= \exp\left(i\left(i\frac{\pi}{2}\right)\right)$

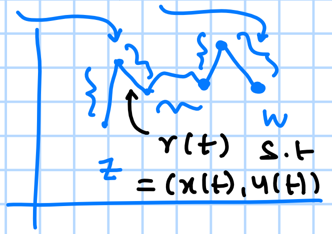
$= \exp\left(-\frac{\pi}{2}\right)$

$P.V(i^i) = e^{-\pi/2}$

Motivation for next week:

path integrals on $\mathbb{C}$:

pieceise smooth curves



$\gamma(t)$ s.t $\gamma(0) = z$
$= (x(t), y(t))$ $\gamma(1) = \omega$

then $\exists\ 0 < t_1 < \cdots < t_n = 1$
s.t $\gamma(t)$ is real-diff on $t_i < t < t_{i+1}$

$\Rightarrow$ length $= \int |\gamma'(t)|\,dt$

total length $= \sum_{i=0}^{n} \int_{t_i}^{t_{i+1}} |\gamma'(t)|\,dt$ where $t_0 = 0 < t_1 < \cdots < t_n = 1$

Incase $f : \mathbb{C} \to \mathbb{C}$ is such that, $\exists F : \mathbb{C} \to \mathbb{C}$ with $F'(z) = f(z)$
then

$$\int_0^1 f(\gamma(t))\,\gamma'(t)\,dt = F(\gamma(1)) - F(\gamma(0))$$

proved using fudanetal theorm of calculus

$$\frac{\partial}{\partial t}(F(\gamma(t))) = f(\gamma(t))\,\gamma'(t)$$

$\Rightarrow$ LHS $= \left(\int \frac{\partial}{\partial t}\left(\quad \right)\right) =$ RHS

This proves that if $f'(z) = 0$ $\forall z \in \mathbb{C}$ then f is a constant as $z \in \mathbb{C}$, $z_0 \in \mathbb{C}$ then if $\gamma(t)$ joins z_0 & z

$$f(z) - f(z_0) = \int_0^1 f'(\gamma(t))\gamma'(t)\, dt$$

for some paramatisation $\gamma(t)$

$$= \int_0^1 (0)\gamma'(t)\, dt$$

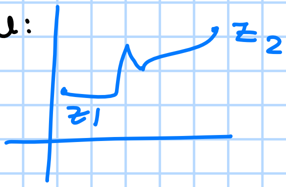
$$f(z) - f(z_0) = 0$$

$\Rightarrow f(z)$ is const function

(This is one more proof for $f'(z) = 0 \Rightarrow f(z)$ is const)

28^{th} Jan :

path integrals in $\mathbb{C}$:

Recall: 

path will be piecewise - smooth

$\gamma : [0,1] \to \mathbb{C}$ s.t

$\gamma(0) = z_1$
$\gamma(1) = z_2$

$\{\gamma(w_i)\}_{i=1}^{n}$ are s.t the path b/w $\gamma(w_i)$ and $\gamma(w_{i+1})$ is smooth

then we can choose γ s.t

$\gamma'(t)$ exist for $t \in (w_i, w_{i+1})$

last time: function $f : \Omega \to \mathbb{C}$

Ω → open set in $\mathbb{C}$

say $\exists F : \Omega \to \mathbb{C}$ s.t

$F'(z) = f(z) \quad \forall z \in \Omega$

where f is cont

then, for a path $\gamma : [0,1] \to \Omega$ joining z_1 and $z_2 \in \Omega$

we have

$\gamma(z_1) = 0$
$\gamma(z_2) = 1$

$$F(z_2) - F(z_1) = \int_0^1 f(\gamma(t))\gamma'(t)\,dt$$

$F(z_2) - F(z_1) = \int_{z_1}^{z_2} f(\gamma(t))\gamma'(t)\,dt$

this gave another proof of $f'(z) = 0$
$\Rightarrow f$ is constant

let $z \in \mathbb{C}$, $z_0 \in \mathbb{C}$
define $\gamma(t)$ to be s.t $\gamma(0) = z_0$
$\gamma(1) = z$

$\gamma(t) = (1-t)z_0 + tz$ → line joing both

$$f(z) - f(z_0) = \int_0^1 f'(\gamma(t))\gamma'(t)\,dt = 0$$

Def : If $\gamma_1 : [0,1] \to \mathbb{C}$, $\gamma_2 : [s_1, s_2] \to \mathbb{C}$ $\}$ smooth curves parameterisation of a smooth curve (one-one corr b/w $[0,1]$ and path)

(same curve)

defines same curve on $\mathbb{C}$

then $\exists$ bijection

$B : [0,1] \to [s_1, s_2]$

s.t B is diff on $(0,1)$

claim: By using def of same curve we want to show if γ_1, γ_2 are same

then $\int_{z \in \gamma_1} f(z)\,dz = \int_{z \in \gamma_2} f(z)\,dz$ → two line integrals are same where $\displaystyle\int_{z\in\gamma_1} f(z)\,dz = \int_0^1 f(\gamma_1(t))\,\gamma_1'(t)\,dt$

proof : here $\gamma_2(s) = \gamma_1(B(t))$
as $\exists\, B : [0,1] \to [s_1, s_2]$

$\gamma_2'(s) = \gamma_2'(B(t))\,B'(t)$ $\quad (\gamma_2(s) = \gamma_2(B(t)))$

$$\int_{s_1}^{s_2} f(\gamma_2(s))\,\gamma_2'(s)\,ds = \int_0^1 f(\underbrace{\gamma_2 B(t)}_{\gamma_1(t)})\,\underbrace{\gamma_2'(B(t))\,B'(t)}_{\gamma_1'(t)}\,dt$$

as $s = B(t)$

$$= \int_0^1 f(\gamma_1(t))\,\gamma_1'(t)\,dt$$

so $\displaystyle\int_{z\in\gamma_1} f(z)\,dz = \int_{z\in\gamma_2} f(z)\,dz$

Note: This means that path integral is independent of the smooth parametrization

Goal of next classes:

$\gamma : [0,1] \to \mathbb{C}$

If γ is a smooth curve s.t $\gamma(0) = \gamma(1)$ then γ is called a closed curve

If $f : \Omega \to \mathbb{C}$ is s.t $\exists\, F : \Omega \to \mathbb{C}$ with $F'(z) = f(z)\ \forall z \in \Omega$

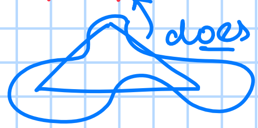
then $\displaystyle\int_{z\in\gamma} f(z)\,dz = F(\gamma(1)) - F(\gamma(0)) = 0$

diff at any point $z \in \Omega$

goal: show that if $f : \Omega \to \mathbb{C}$ is holomorphic, then $\exists F : \Omega \to \mathbb{C}$ s.t $F'(z) = f(z)\ \forall z \in \Omega$

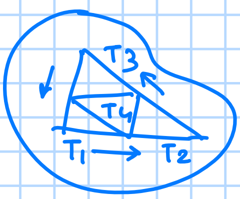
Theorm: (Goursat's theorm) If $f : \Omega \to \mathbb{C}$ is holomorphic & $\text{int}(T) \subseteq \Omega$ is a triangle inside Ω, then

$$\int_T f(z)\,dz = 0$$



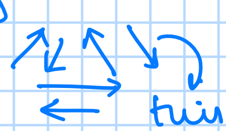
proof:

$T_0 = T$



$$\int_{T_0} f(z)\,dz = \int_{T_1(1)} f(z)\,dz + \int_{T_1(2)} f(z)\,dz + \int_{T_1(3)} f(z)\,dz + \int_{T_1(4)} f(z)\,dz$$

as



this line integral cancels the inside one

$$\Rightarrow \left|\int_{T_0} f(z)\,dz\right| \le \left|\int_{T_1(1)} f(z)\,dz\right| + \cdots$$

say d_0 = diameter of T_0
p_0 = perimeter of T_0

notice : $\operatorname{diam}(T_1(j)) = \frac{1}{2} d$
$\operatorname{per}(T_1(j)) = \frac{1}{2} p_0$

say T_1 = triange where
$\left| \int_{T_1(j)} \dots \right|$ is the biggest

Iteration: at n^{th} step:

$$\int_{T_n} f(z)\,dz = \sum_{j=1}^{4} \int_{T_{n+1}(j)} f(z)\,dz$$

$$\Rightarrow \left| \int_{T_n} f(z)\,dz \right| \leq 4 \left| \int_{T_{n+1}} f(z)\,dz \right|$$

$$\Rightarrow \left| \int_{T_0} f(z)\,dz \right| \leq 4^n \left| \int_{T_n} f(z)\,dz \right|$$

$\operatorname{diam}(T_n) = 2^{-n} d$
$\operatorname{perm}(T_n) = 2^{-n} p$

obtain $\mathcal{T}_1$ = convex hull(T_1) (smallest convex polygon containg T_1)

$\mathcal{T}_0 \supseteq \mathcal{T}_1 \supseteq \mathcal{T}_2 \dots\dots$
and $\lim_{n \to \infty} d_n = 0$

$$\Rightarrow \bigcap_{i=0}^{\infty} \mathcal{T}_i = \{z_0\}$$

Say, $z \in \Omega$ then $f(z) = f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0)$ (Taylor expansion)
where $\lim_{z \to z_0} \psi(z) = 0$

$$\int_{T_n} f(z)\,dz = \int_{T_n} \Big(f(z_0) + f'(z_0)(z - z_0) + \psi(z)(z - z_0) \Big)\,dz$$

$$= \int_{T_n} \underbrace{f(z_0)}_{\text{const}}\,dz + \int_{T_n} \underbrace{f'(z_0)(z - z_0)}\,dz + \int_{T_n} \psi(z)(z - z_0)\,dz$$

$F(z) = z f(z_0)$
then $F' = f$

$g(z) = f'(z_0)\frac{(z - z_0)^2}{2}$
$g'(z) = f'(z_0)(z - z_0)$

$$= 0 + 0 + \int_{T_n} \psi(z)(z - z_0)\,dz$$

$$\Rightarrow \left| \int_{T_n} f(z)\,dz \right| \leq \int_{T_n} |\psi(z)||z-z_0||dz| \leq \sup_{z\in \gamma_n} |\psi(z)| . d_n . p_n = \sup_{z\in\gamma_n} |\psi(z)| 2^{-n} d_0 \, 2^{-n} p_0$$

$$\int_{T_n} |dz| = \text{length of } T_n$$

$$\Rightarrow \left| \int_{T_0} f(z)\,dz \right| \leq 4^n \left| \int_{T_n} f(z)\,dz \right| < \cancel{4^n} \sup_{z\in\gamma_n} |\psi(z)| \cancel{2^{-n}} d_0 \cancel{2^{-n}} p_0 = d_0 p_0 \sup_{z\in\gamma_n} |\psi(z)|$$

$$\Rightarrow \left| \int_{T_0} f(z)\,dz \right| \leq d_0 p_0 \sup_{z\in\gamma_n} |\psi(z)| < d_0 p_0 \left(\frac{\varepsilon}{d_0 p_0} \right) \text{ for large } n$$

$$\Rightarrow \forall \varepsilon > 0 \quad \left| \int_{T_0} f(z)\,dz \right| < \varepsilon$$

$$\Rightarrow \left| \int_{T_0} f(z)\,dz \right| = 0$$

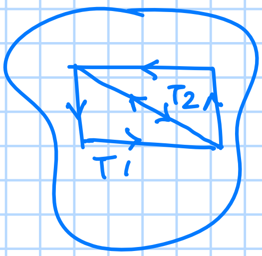
$$\Rightarrow \int_{T_0} f(z)\,dz = 0$$

Note: If R is a rectangle s.t

$R \cup \text{int}(R) \subseteq \Omega$

& f is hol on Ω then:

$$\int_R f(z)\,dz = 0$$



$$\int_{T_1} f(z)\,dz + \int_{T_2} f(z)\,dz = \int_R f(z)\,dz$$

($\int_{T_1} f(z)\,dz \to 0$, $\int_{T_2} f(z)\,dz \to 0$)

$$\Rightarrow \int_R f(z)\,dz = 0$$

This can be generalised even further

31^{st} Jan :

recap :

If f is holomorphic in some $\underset{\text{open}}{\Omega} \subseteq \mathbb{C}$ that contains a Δ and $\mathrm{int}(\Delta)$

then $\int_{\Delta} f(z)\,dz = 0$ (Goursat's theorem)

Now if f is s.t $\exists F : \Omega \to \mathbb{C}$ satisfying $F'(z) = f(z)$ then $\int_{\gamma} f(z)\,dz = 0$ if γ is a closed curve. (FTC)

goal-1: If f is holomorphic on some $\underset{\text{open}}{\Omega} \subseteq \mathbb{C}$, then f is infinitely diff in Ω

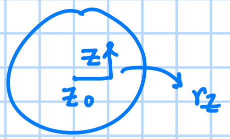
goal-2: If f is hol at z then $f(z) = \frac{1}{2\pi i} \oint_{C} \frac{f(\zeta)}{\zeta - z}\, d\zeta$

C ← some circle s.t $z \in \mathrm{int}(\text{circle})$

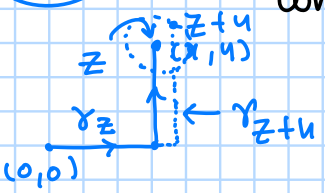
Theorem : If D is a disc in $\mathbb{C}$ s.t f is holomorphic in $\mathrm{int}(D)$ then
$D = \{z \mid |z - z_0| \le R\}$
then f has a primitive in $\mathrm{int}(D)$

($\exists F : \mathrm{int}(D) \to \mathbb{C}$ s.t $F'(z) = f(z)$ $\forall z \in \mathrm{int}(D)$)

proof : WLOG : centre is O or $z_0 = 0$

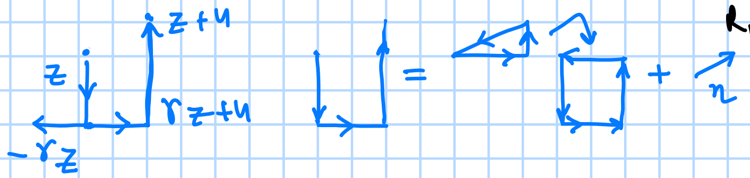


$$F(z) = \int_{\gamma_z} f(\omega)\,d\omega$$



consider : $F(z+h) - F(z)$ where $h \in \mathbb{C}$ is st $z + h \in \mathrm{int}(D)$

$$F(z+h) - F(z) = \int_{\gamma_{z+h}} f(\omega)\,d\omega - \int_{\gamma_z} f(\omega)\,d\omega$$



$$= \int_{R,\,\text{anticlock}} f(\omega)\,d\omega + \int_{T,\,\text{anticlock}} f(\omega)\,d\omega + \int_{\eta} f(\omega)\,d\omega$$

η ← straight line from z to $z+h$

$$= \int_{\eta} f(\omega)\,d\omega$$

now as f is holomorphic $\Rightarrow f$ is cont

$\Rightarrow f(\omega) = f(z) + \psi(\omega)$ s.t $\lim_{\omega \to z} \psi(\omega) = 0$

$$F(z+h) - F(z) = \int_{\eta} \left(f(z) + \psi(\omega)\right) d\omega$$

$$= f(z)\int_{\eta} 1\, dw + \int_{\eta} \psi(w)\, dw$$

$$F(z+h) - F(z) = f(z)(h) + \int_{\eta} \psi(w)\, dw$$

$$\Rightarrow \frac{F(z+h)-F(z)}{h} = \frac{f(z)\cdot h}{h} + \int_{\eta} \frac{\psi(w)\, dw}{h}$$

now $\left| \int_{\eta} \frac{\psi(w)\, dw}{h} \right| \leq \frac{1}{h} \int_{\eta} |\psi(w)|\,|dw|$

$$\leq \frac{1}{\cancel{h}} \underbrace{\sup_{w \in [z, z+h]^{\circ}}}_{\text{straight path joing both}} |\psi(w)| . \cancel{h} \longrightarrow 0 \text{ as } h \to 0$$

so $\lim_{h \to 0} \frac{F(z+h) - F(z)}{h} = f(z) = F'(z)$

<u>Collorary</u>: If f is hol on $\text{int}(D)$, then $\oint_{\gamma} f(z)\,dz = 0$ if γ is a closed path inside $\text{int}(D)$

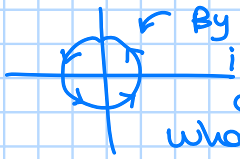
<u>Note</u>: If f is hol inside $\Omega \subseteq \mathbb{C}$ s.t Ω contains a disk D, then $\exists\, F: D \to \mathbb{C}$ s.t $F'(z) = f(z)\ \forall z \in D$

This works for <u>D closed</u>, as D is closed, Ω is open, choose another disc D' s.t $D \subsetneq \text{int}(D')$ and $D' \subseteq \Omega$

<u>Theorm</u>: (Cauchys integral formula) If f is hol on $\Omega \subseteq \mathbb{C}$ (open) s.t Ω contains a disk D (say $C = \partial D$) then $\forall z \in \text{int}(D)$

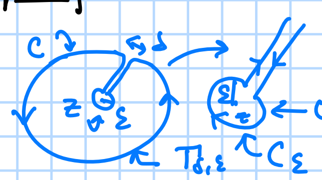
$$\frac{1}{2\pi i} \oint_C \frac{f(w)}{w - z}\, dw = f(z)$$

<u>local information</u>	<u>global information</u>
$f(z)$ and the fact that f is diff in a neighb of z	integral over C



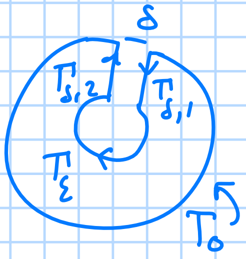
By somehow integrating over the circle we get idea of what is inside it

<u>proof</u>:



This is called a keyhole contour

Firstly there is a bigger open disk cont C, so $\exists\, F: \Omega \to \mathbb{C}$ s.t $F'(z) = f(z)$



now as $\Gamma_{\delta,\varepsilon}$ is a closed path:

$$\int_{\Gamma_{\delta,\varepsilon}} f(w)\,dw = 0$$

$$\int_{\Gamma_0} f(w)dw + \int_{\Gamma_{\delta,1}} f(w)dw + \int_{\Gamma_{\delta,2}} f(w)dw + \int_{\Gamma_\varepsilon} f(w)dw = 0$$

if $\delta \to 0$, $\varepsilon \to 0$

$\Rightarrow \Gamma_0 \to C$, $\Gamma_\varepsilon \to C_\varepsilon$, by taking limit: $\delta \to 0$

$$0 = \int_C f(w)dw + 0 + \int_{\Gamma_\varepsilon} f(w)dw$$

now $\displaystyle\int_{\Gamma_{\delta,\varepsilon}} \frac{f(w)}{w-z}\,dw = 0$

$\frac{f(w)}{w-z}$ → this is holomorphic in $\Omega \setminus \{$a closed disk cont $z\}$

$$\Rightarrow 0 = \int_{\Gamma_0} \frac{f(w)}{w-z}dz + \int_{\Gamma_{\delta,1}} \frac{f(w)}{w-z}dw + \int_{\Gamma_{\delta,2}} \frac{f(w)}{w-z}dw + \int_{\Gamma_\varepsilon} \frac{f(w)}{w-z}dw$$

now $\delta \to 0$

$$0 = \int_C \frac{f(w)}{w-z}dz + \int_{\Gamma_\varepsilon} \frac{f(w)}{w-z}dz$$

Paramatrise Γ_ε:

$\Gamma_\varepsilon(t) = \underset{\text{centre}}{z} + \underset{\text{radius}}{\varepsilon} e^{-it}$ and let $t \in (0, 2\pi)$

$$\Rightarrow 0 = \int_C \frac{f(w)}{w-z}dw + \int_0^{2\pi} \frac{f(z+\varepsilon e^{it})}{\varepsilon e^{-it}}(-\varepsilon i e^{-it})dt$$

$$\Rightarrow i\int_0^{2\pi} \frac{f(z+\varepsilon e^{it})\,\cancel{\varepsilon e^{-it}}}{\cancel{\varepsilon e^{-it}}}dt = \int_C \frac{f(w)}{w-z}dw$$

for $\varepsilon \to 0$

$$\Rightarrow 2\pi i\, f(z) = \int_C \frac{f(w)}{w-z}dw$$

$$\Rightarrow f(z) = \frac{1}{2\pi i}\int_C \frac{f(w)}{w-z}dw$$

$$\left(\text{thus } f(z) = \frac{1}{2\pi i}\oint_C \frac{f(w)}{w-z}dw\right)$$

Cauchy's integral formula:

If f is hol on $\underset{\text{open}}{\Omega} \subseteq \mathbb{C}$ then f is ∞-diff on Ω

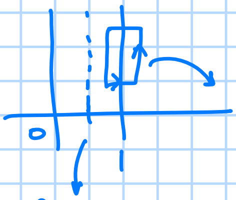
$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(w)}{(w-z)^{n+1}}\, dw$$

(this is what we want)

where C is a counterclockwise curve $C \subseteq \Omega$ and $z \in \operatorname{int} D$

Note: If f is "meromorphic" then $\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)}\, dz$ $= \#\text{zeros} - \#\text{poles}$ of f inside $\operatorname{int}(D)$

$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}$



$\frac{1}{2\pi i} \int_C \frac{\zeta'(s)}{\zeta(s)}\, d$

$\operatorname{Re}(s) = \frac{1}{2}$

(this is extra but good example of application)

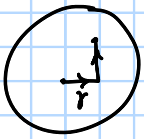
4th feb:

recap: If $f:\Omega \to \mathbb{C}$ is holomorphic s.t

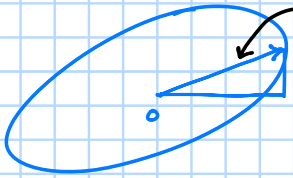
$\underset{\text{open}}{\Omega} \subseteq \mathbb{C}$ is an open disc then

$\exists\, F:\Omega \to \mathbb{C}$ s.t

$F'(z) = f(z)$

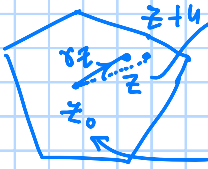


now, if $\underset{\text{open}}{\Omega} \subseteq \mathbb{C}$ is some convex set then this construction may be somewhat problematic



this is good → use goursat's theorm and use triangles

← not good

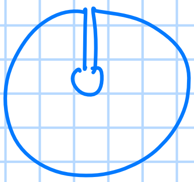


$\forall z$

any point

Note: Now for any open convex set we have $\exists F$ s.t $F'(z) = f(z)$

Recap: keyhole contour



$\underset{\text{open}}{\Omega} \in \mathbb{C}$

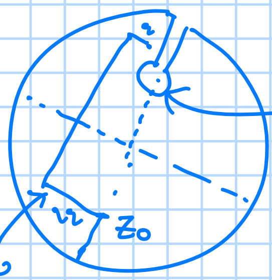
$\exists\, D \subseteq \Omega$ and

f is hol on Ω

then $\displaystyle \frac{1}{2\pi}\int_C \frac{f(w)}{w - z}\,dw = f(z)$

$\forall z \in \text{int}(D)$

Note: keyhole contour is not convex



not holomorphic here

or

never intersects on z_0

new path construction to make sure $F'(z) = f(z)$

Theorm: If $f:\Omega \to \mathbb{C}$, $\underset{\text{open}}{\Omega} \subseteq \mathbb{C}$ s.t f is hol on Ω, and say Ω is s.t $\exists$ closed disc $D \subseteq \Omega$ then f is inf diff on D.

and $\displaystyle f^{(n)}(z) = \frac{n!}{2\pi i}\oint_C \frac{f(w)}{(w-z)^{n+1}}\,dw \quad \forall z \in \text{int } D$, where

$D \subsetneq D' \subseteq \Omega$ and $C = \partial D'$

proof : By induction for $n=0$

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\omega)}{\omega - z}\,d\omega$$

if $$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i}\oint_C \frac{f(\omega)}{(\omega-z)^n}\,d\omega$$

$$f^{(n)}(z) = \lim_{h\to 0} \frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h}$$

if it exist

where $|h|$ is small

enough for $z+h \in \mathrm{int}(D')$

$$\lim_{h\to 0} \frac{(n-1)!}{2\pi i}\frac{1}{h}\oint_C f(\omega)\left[\frac{1}{(\omega-z-h)^n} - \frac{1}{(\omega-z)^n}\right]d\omega$$

$A^n - B^n = (A-B)(A^{n-1} + A^{n-2}B + \cdots + B^{n-2}A + B^{n-1})$

take

$A = (\omega - z)$ $\quad B = (\omega - z - h)$

$$\lim_{h\to 0}\frac{(n-1)!}{(2\pi i h)}\oint_C \frac{f(\omega).\overbrace{h}^{A-B}[A^{n-1} + \cdots + B^{n-1}]}{(\omega-z-h)^n(\omega-z)^n}\,d\omega$$

$$= \lim_{h\to 0}\frac{(n-1)!}{(2\pi i)}\oint_C f(\omega)\frac{[A^{n-1}+\cdots+B^{n-1}]}{(A)^n(B)^n}\,d\omega$$

$$= \frac{(n-1)!}{2\pi i}\oint_C \lim_{h\to 0} f(\omega)\frac{[A^{n-1}+\cdots\cdots+B^{n-1}]}{(A)^n(B)^n}\,d\omega$$

$\left(\lim_{h\to 0}\int = \lim_{h\to 0}\right.$ dominated convergence theorem$\left.\right)$

$$= \frac{(n-1)!}{2\pi i}\oint_C f(\omega)\frac{(n)(A^{n-1})}{(A)^n(A^n)}\,d\omega$$

$$= \frac{(n)!}{2\pi i}\oint_C \frac{f(\omega)}{A^{n+1}}\,d\omega$$

If f is hol on $\Omega \subseteq \mathbb{C}$ cont closed disk $D \subseteq \Omega$ and centre$(D) = z_0$
open $\quad$ rad$(D) = R$
$C = \delta D$

then $$|f^{(n)}(z_0)| \leq n!\frac{\|f\|_C}{R^n}$$

where $$\|f\|_C = \sup_{z\in C}|f(z)|$$

as $$|f^{(n)}(z_0)| \leq \left|\frac{n!}{2\pi i}\right| \int_C \left|\frac{|f(\omega)|}{(\omega-z_0)^{n+1}}\right| |d\omega|$$

$$\leq \frac{n!}{2\pi}\sup_{\omega\in C}|f(\omega)| \times \frac{2\pi R}{|R|^{n+1}} \quad \text{as } |\omega - z_0| = |R|$$

$$= \frac{n!}{R^n} \underbrace{\sup_{w \in C} |f(w)|}_{\|f\|_C}$$

so $$|f^{(n)}(z_0)| \leqslant \frac{n!}{R^n} \|f\|_C$$

Theorm :(Liouville's theorm) If $f : \mathbb{C} \to \mathbb{C}$ is holomorphic and bounded (meaning $\exists B \geqslant 0$ s.t $|f(z)| \leqslant B \ \forall z \in \mathbb{C}$) then f is const

proof : let's show $f'(z) = 0 \ \forall z \in \mathbb{C}$

let $z \in \mathbb{C}$, D be disk of centre z, radius R

then
$$|f'(z)| \leqslant \frac{n! \|f\|_C}{R} \qquad n = 1$$

$$= \frac{\|f\|_C}{R} \leqslant \frac{B}{R} \text{ given}$$

letting $R \to \infty$
$$\Rightarrow |f'(z)| \leqslant 0$$
$$\Rightarrow f'(z) = 0$$

Theorm : (Fundamental theorm of algebra) A non-constant polynomial of degree n ($n \geqslant 1$) has exactly n many roots (counting multiplicity)

proof : By contradiction

$$P(z) = a_0 + a_1 z + \cdots + a_n z^n$$
$$n \geqslant 1 \quad a_n \neq 0$$

Step-1 : $P(z)$ has atleast one root

if not then $\frac{1}{P(z)}$ is bounded

$$\frac{P(z)}{z^n} = \frac{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1} + a_n z^n}{z^n}$$

$$= a_n + \left(\frac{a_0}{z^n} + \cdots + \frac{a_{n-1}}{z} \right)$$

consider a big disc D of centre 0 radius R

then for $z \in \partial D$
$$|z| = R$$

$$\left| \frac{a_0}{z^n} + \cdots \frac{a_{n-1}}{z} \right| \leqslant (n) \frac{\max |a_k|}{R}$$

letting R to be huge $|a_n| > 2n \frac{\max |a_k|}{R}$

$$\Rightarrow |P(z)| / |z^n| \geqslant |a_n| / 2 \text{ for } z \in \partial D$$

$\frac{|P(z)|}{|z^n|}$ $\quad |z| > R$ then

$$\frac{|P(z)|}{|z^n|} = \left|\frac{a_0}{z^n} + \cdots \frac{a_{n-1}}{z} + a_n\right|$$

$$> \frac{|a_n|}{2}$$

$\Rightarrow |P(z)| \geq \frac{|a_n||z^n|}{2} = \frac{|a_n|R^n}{2} \quad \forall z \in \mathbb{C}\setminus D$

as $|P(z)| \geq |a_n z^n| \frac{1}{2} \geq \frac{|a_n||z^n|}{2}$

$\Rightarrow \frac{1}{|P(z)|} \leq \frac{2}{|a_n|R^n}$ for $z \in \mathbb{C}\setminus D$

and $|P(z)| \leq B$ for some $B \in \mathbb{R}$ for $z \in D$

and

$\frac{1}{P(z)}$ does not vanish

$\Rightarrow \frac{1}{P(z)}$ is diff for $z \in D$

$\Rightarrow \exists B_1 \in \mathbb{R}_{>0}$ s.t

$\frac{1}{|P(z)|} < B_1$ for $z \in D$

$\Rightarrow \frac{1}{|P(z)|} \leq \max\left(B_1, \frac{2}{|a_n|R^n}\right) \leq R_2$

$\forall z \in \mathbb{C}$

$\Rightarrow \frac{1}{P(z)}$ is const $*$

now $P(\omega_1) = 0$ for some $\omega_1 \in \mathbb{C}$

$P(z) = b_n(z-\omega_1)^n + b_{n-1}(z-\omega_1)^{n-1} + \cdots b_0$

$\Rightarrow P(\omega_1) = b_0 = 0$

$\Rightarrow P(z) = (z-\omega_1)\underbrace{(b_n(z-\omega)^{n-1} + \cdots b_0)}_{\deg(n-1)}$

By reduction
$P(z)$ has exactly
n roots

5th Feb:

Recap: $\lim_{h\to 0} \frac{(n+1)!}{2\pi i}\int_C f(\omega)\left(\frac{1}{A^n} - \frac{1}{B^n}\right)d\omega$

$A = (z-\omega-h)$
$B = (z-\omega)$

now, $\{g_n(x)\}_{n\geq 1}$ converges uniformly to g on a closed set $\Omega \subseteq \mathbb{R}$

$g_n : \mathbb{R} \to \mathbb{C}$ then

$$\lim_{n\to\infty}\int_\Omega g_n(x)dx = \int_\Omega g(x)dx$$

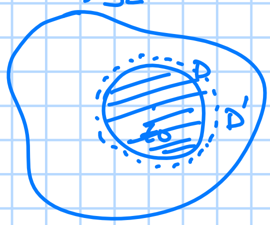
$$\frac{1}{(z-\omega-h)^n} \longrightarrow \frac{1}{(z-\omega)^n}$$

uniformly as $h \to 0$

Theorem: (Power series expansion) If $f:\Omega \to \mathbb{C}$ ($\Omega \subseteq \mathbb{C}$ open) is hol and $D \subseteq \Omega$ (closed disk) then $f(z)$ has a power series expansion inside D with centre of power series = centre of D.

proof: given $D \subseteq \Omega$, find $D' \supsetneq D$, $D' \subseteq \Omega$ say centre(D) = centre$(D') = z_0$

take $z \in \mathrm{int}(D')$ (this is $z \in D \subseteq \mathrm{int}\,D'$) Ω



$$f(z) = \frac{1}{2\pi i}\oint_{\delta D'} \frac{f(\omega)}{\omega - z}d\omega$$

$$\frac{1}{\omega - z} = \frac{1}{(\omega - z_0) - (z - z_0)}$$

$$= \frac{1}{\omega - z_0} \times \frac{1}{1-\left(\frac{z-z_0}{\omega - z_0}\right)}$$

$\omega \in \delta D'$ and $z \in \mathrm{int}(D')$

$\Rightarrow |z-z_0| < |w - z_0|$

$\Rightarrow \frac{|z-z_0|}{|\omega - z_0|} < 1$

so $$\frac{1}{1-\left(\frac{z-z_0}{\omega-z_0}\right)} = \sum_{n=0}^{\infty}\left(\frac{z-z_0}{\omega-z_0}\right)^n \quad \left(\frac{1}{1-r} = 1 + r + r^2 + \dots\right)$$

$$\Rightarrow f(z) = \frac{1}{2\pi i}\int_{\delta D'}\frac{f(\omega)}{(\omega - z_0)}\sum_{n\geq 0}\left(\frac{z-z_0}{w-z_0}\right)^n d\omega$$

$$\underset{\substack{\uparrow\\ \text{unique conv of geom series}}}{=} \sum_{n\geq 0}\Bigg[\underbrace{\frac{n!}{2\pi i}\int_{\delta D'}\frac{f(\omega)\,d\omega}{(\omega - z_0)^{n+1}}}_{f^{(n)}(z_0)}\Bigg]\frac{(z-z_0)^n}{n!}$$

$$f(z) = \sum_{n\geq 0}\frac{f^{(n)}(z_0)}{n!}(z-z_0)^n$$

$\therefore$ for $z \in \mathrm{int}(D')$ $(\forall z \in D)$ we can write it as power series

Differentiable: $\frac{\partial}{\partial z} f(z)$ exist at z_0

$\Rightarrow$ diff at z_0

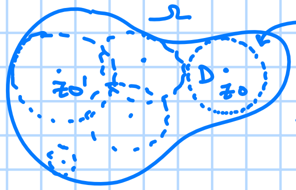
Analytic: f is analytic at $z = z_0$ if $\exists$ a (open) neighbourhood $U \ni z_0$ s.t f has a power series expansion in U

Holomorphic: Analytic funtion $f : \mathbb{C} \to \mathbb{C}$

Note: we will use all three interchangibly

$$g(z) = \sum_{n \geq 0} a_n (z - z_0)^n \qquad |z - z_0| < R, \; R > 0$$

then
$$a_n = \frac{g^{(n)}(z_0)}{n!}$$
by diff $g(z)$ n-times



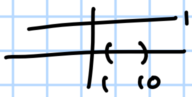
$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(z_0)(z - z_0)^n}{n!}$$

not all points

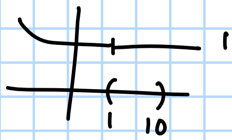
Rigidity:

Say $g : \mathbb{R} \to \mathbb{R}$ s.t g is ∞-diff on $\mathbb{R}$. Then if $\Omega \subseteq \mathbb{R}$ open connected then $g(\Omega)$ does not determine g

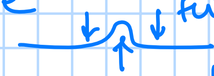
$$g : \mathbb{R} \to \mathbb{R}, \quad x \mapsto 1$$



but

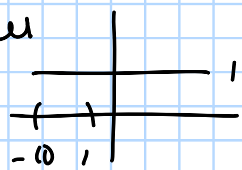


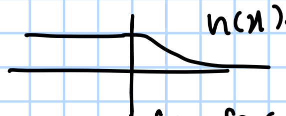
($g : \mathbb{R} \to \mathbb{R}$ is not rigid i.e



this does not merge)

even





$$h(x) = \begin{cases} e^{-x^2} ; & x \geq 0 \\ 1 ; & x < 0 \end{cases}$$

So for $(-10, 1)$ we can have so many funtion which are ∞-diff

$\therefore g$ is not very rigid

Theorm: let f be hol on $\Omega \subseteq \mathbb{C}$ which is open & connected, say that $\exists$ seq $\{w_k\}_{k \geq 1} \subseteq \Omega$ s.t each $f(w_k) = 0$ $\forall k \geq 1$ and $\{w_k\}_{k \geq 1}$ has a lim point $z_0 \in \Omega$ then
$$f \equiv 0 \text{ on } \Omega$$

this is an example which we can prove by using this theorm.

D $f(z) = 0 \; \forall z \in D \Rightarrow f \equiv 0$ on Ω (this mean $f : \mathbb{C} \to \mathbb{C}$ is rigid $\Rightarrow$)

proof: Step 1: $f \equiv 0$ in a small nbd of z_0

since $z_0 \in \Omega$, $\exists$ disk $D_{z_0} \subseteq \Omega$ centred $\uparrow$ closed

s.t $D \subseteq \Omega$

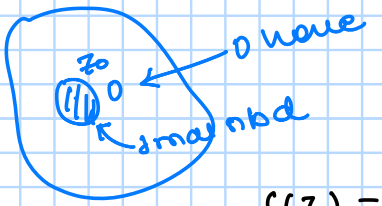
let $z \in \text{int}(D)$ and expand $f(z)$

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n$$

if f is not identically 0 on $\text{int}(D)$ then

$\exists m \geq 0$ s.t $a_m \neq 0$

$\downarrow$ choose s.t m is smallest of such kind



$$f(z) = a_m (z - z_0)^m \left(1 + g(z - z_0)\right)$$

series centred at z_0

as $a_m \neq 0$

where $g(0) = 0$

as $g(0) \Rightarrow z - z_0 = 0$

$\Rightarrow z = z_0$

Since $\{w_k\}$ have z_0 as limit points choose

$$\{w_{k_i}\}_{k_i} \subseteq \{w_k\} \text{ s.t } w_{k_i} \neq z_0$$

subseq

s.t $\lim_{i \to \infty} w_{k_i} = z_0$

since $z_0 \in \text{int}(D)$ then $\exists K$ s.t $k_i > K$

$\Rightarrow w_{k_i} \in \text{int}(D)$

$$\Rightarrow f(w_{k_i}) = a_m (w_{k_i} - z_0)^m \left[1 + g(w_{k_i} - z_0)\right]$$

$$\Rightarrow 0 = f(w_{k_i}) = \underset{\neq 0}{a_m} \underset{\neq 0}{(w_{k_i} - z_0)^m} \left[\underbrace{1 + g(w_{k_i} - z_0)}_{\neq 0}\right]$$

as $k_i \to \infty$

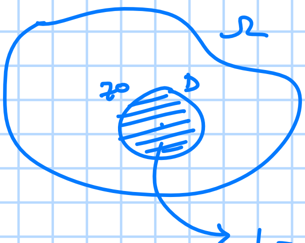
$g(w_{k_i} - z_0) \to 0$

$1 + g(w_{k_i} - z_0) \to 1$

$\therefore$ not zero

$\Rightarrow$ this is a contradiction

$\Rightarrow f \equiv 0$ on D



$f \equiv 0 \ \forall z \in D$

let $U = \text{int}\{z \in \Omega \mid f(z) = 0\}$

$D \subseteq U$ and U is open by defination and also by defination non-empty

if $\{z_n\} \subseteq U$ is cauchy sequence in U then $\lim_{n \to \infty} z_n \in \Omega$

and $f(z_n) = 0 \;\forall\; n \geq 1$

($\{z_n\}$ is a cauchy in C then limit point z_0 has a ball around it $\Rightarrow$ $z_0 \in \Omega$ and in int)

$\Rightarrow \lim_{n\to\infty} f(z_n) = f(\lim_{n\to\infty} z_n)$

$\qquad \parallel$

$\lim_{n\to\infty} 0 = 0$

$\Rightarrow f(\lim_{n\to\infty} z_n) = 0$

$\{z_n\}$ is a sequence which has a limit point z

$f(z_0) = 0 \;\forall n \in \mathbb{Z}$

& $\{z_n\}, z \in \Omega$

$\Rightarrow \exists$ open neighbourhood of z s.t $f \equiv 0$ in that neighbourhood

$\Rightarrow \lim_{n\to\infty} z_n \in U$ as $\exists$ open nbd s.t $z \in$ that open nbd

$\Rightarrow$ any cauchy in U has limit point in U

$\Rightarrow U$ is closed

now U (in subspace topology Ω)

$U \underset{\text{open}}{\subseteq} \Omega$

$U \underset{\text{closed}}{\subseteq} \Omega \Rightarrow V = \Omega \setminus U \underset{\text{open}}{\subseteq} \Omega$

(as U is int, U is open, U is also closed, $U \neq \phi \Rightarrow U = X$)

$U \cap V = \phi$
$U \cup V = \Omega$ } this contradicts connectedness unless $V = \phi$

$\Rightarrow U = \Omega \setminus V = \Omega$

(A, B of X are sep if $A \cap \bar{B} = \phi$ and $\bar{A} \cap B = \phi$

$E \subseteq X$ is connected if E is not a union of non-empty sepsets)

Here U is both open and close

$\Rightarrow U = \bar{U}$

$V = \bar{V}$

and $U \cap \bar{V} = U \cap V = \bar{U} \cap V = \phi$

so U, V are not connected

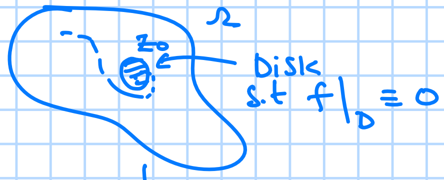
but $\Omega = U \cup V$ $\quad *$

Ω connected; $U \cup V$ union of two not connected sets

7^{th} feb:

Recap: If $\{z_n\}$ is a sequence of distinct points s.t $z \in \Omega$ is its limit point and $f(z_n)=0 \ \forall n \geq 1$ then $f \equiv 0$ on Ω

$\Omega \leftarrow$ open connected subset of $\mathbb{C}$



By connectedness whole zero.

Analytic continuation of holomorphic function:

Existence and uniqueness

say $\underset{\text{open}}{\Omega_2} \subseteq \underset{\text{open}}{\Omega_1} \subseteq \mathbb{C}$ both connected

$f: \Omega_1 \longrightarrow \mathbb{C}$ is hol (on Ω_1)

$F_1, F_2: \Omega_2 \rightarrow \mathbb{C}$ are hol (on Ω_2)

s.t

$F_1(z) = F_2(z) = f(z) \ \forall z \in \Omega_1$

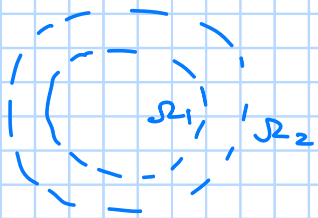
then

$F_1 = F_2$ on Ω_2

$F_1 = F_2$ are called analytic con of f

$\Omega_1 \leftarrow$ need not be connected

(Here Ω_2 should be connected)



$g(z) = F_1(z) - F_2(z)$

then $\forall z \in \Omega_1$

$g(z) \equiv 0$ for Ω_1

$\Rightarrow g(z) \equiv 0$ for Ω_2

$\Rightarrow F_1 = F_2$ on Ω_2

Note: This is the existence part

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} \quad \text{for } Re(s) > 1$$

Riemann: Extended $\zeta(s)$ to all of $\mathbb{C} \leftarrow$ meromorphically

Cauchy: $f^{(n)}(z) = \dfrac{n!}{2\pi i} \displaystyle\int_C \frac{f(w)}{(w-z)^{n+1}} dw$

$f(z) = u(x,y) + i v(x,y)$

where $z = x + iy$

$u, v : \mathbb{R}^2 \rightarrow \mathbb{R}$

$f: \mathbb{C} \rightarrow \mathbb{C}$

f is diff $\Rightarrow U_x = V_y, \ U_y = -V_x$

Note: If $U_x = V_y$, $U_y = -V_x$ true on an open set Ω $\Rightarrow$ f is diff

Note: Now we don't have to check the cont of these U_x, U_y, V_x, V_y as f is C^∞ then

$$f(z) = U(x,y) + i\,V(x,y)$$

partial derivative exist

If f is $\mathbb{C}$-diff
$f = u + iv$
as f is $C^\infty \Rightarrow U, V$ are C^1

Clairaut's theorm / Schwarz theorm:

If $f : \mathbb{C} \to \mathbb{C}$ is $\mathbb{C}$-diff
then $f = u + iv$
$u_{xy}, u_{yx}, v_{xy}, v_{yx}$
are cont

$\Rightarrow U_{xy} = U_{yx}$ & $V_{xy} = V_{yx}$

Exe: Given $U : \mathbb{R}^2 \to \mathbb{R}$ s.t $U \in C^2(\mathbb{R}^2)$
i.e $U_{xx}, U_{xy}, U_{yx}, U_{yy}$ exist & are cont
can we find $f : \mathbb{C} \to \mathbb{C}$ s.t $Re(f) = u$?

$$U_{xx} = (U_x)_x \underset{C-R}{=} (V_y)_x = V_{yx}$$

$$U_{yy} = (U_y)_y \underset{C-R}{=} (-V_x)_y = -V_{xy}$$

$$\Rightarrow U_{xx} + U_{yy} = 0$$

$$\Rightarrow \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)}_{\Delta\ =\ \text{Laplacian}} u = 0.u$$

$$\Delta . u = 0. u$$

Laplacian operator on $C^\infty(\mathbb{R}^2)$
(atleast twice diff)

$u \in C^2(\mathbb{R}^2)$ is called s.t $\Delta u = 0$ then it is called Harmonic function

Note: $\Delta . u = 0. u$ and $u \in C^2(\mathbb{R}^2)$
$\Rightarrow u$ is harmonic function

similarly for $V_{xx}+V_{yy}=0$

Def: (Harmonic conjugates) U, V are called harmonic conjugates if $\Delta . U = 0.$ U $U \in C^2(\mathbb{R}^2)$
$\Delta . V = 0.$ V $V \in C^2(\mathbb{R}^2)$
and $f = u + iv$

we are interested in finding $V: \mathbb{R}^2 \to \mathbb{R}$ given $U: \mathbb{R}^2 \to \mathbb{R}$ which is harmonic?
or to be conjugate
for now if $U \in C^\infty(\mathbb{R}^2)$ then yes
($f = u + iv$ is C-diff)

$$V(x,y) = \int_b^y u_x(x,t)\,dt$$

then U_x is cont
we need V s.t
$V_y = U_x$

for $V(x_0, y_0) = \int_b^{y_0} u_x(x_0, y)\,dy$

then $V_y = u_x$

$$v_y = \frac{\partial}{\partial x}\int_b^{y_0} u_x(x_0,y)\,dy$$

$V_y = u_x$
$V_y = -V_x$
if $V(x,y) = \int_b^y u_x(x,y')\,dy'$
$V_x = \int_b^y \frac{\partial}{\partial x} V_x(x,y')\,dy'$
$= \int_b^y u_{xx}\,dy = \int_b^y -u_{yy}\,dy$
$= -u_y + C$

and check
$V_x = -u_y$

Exe: $u(x,y) = e^x \cos(y)$
$u_x = e^x \cos y$
$u_{xx} = e^x \cos y$

$u_{yy} = -e^x \cos y$

$\Rightarrow u_{xx} + u_{yy} = 0$

now

$v_y = u_x$ $\quad V = \int e^x \cos(y)\,dy$ $\quad V_y = e^x \cos(y) = u_x$

$V = e^x \sin(y) + \phi(x)$
$V_x = e^x \sin(y) + \phi'(x)$
$= -u_y = -(-e^x \sin y)$

$\Rightarrow \phi'(x) = 0$
$\Rightarrow \phi = C$
$f(x,y) = e^x \cos(y) + ie^x \sin(y) + iC$

$f(z) = e^z + iC$

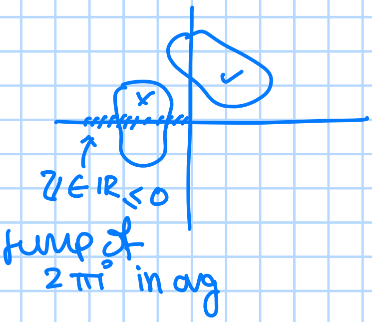
Note: what we want is if $U(x,y)$ given
① is U harmonic
② if U is harmonic then does $\exists\, V$ s.t $\leftarrow$ This is what we are asking
$f = u + iv$
(i.e V is also harmonic and is called harmonic conjugate)

This V is what we want $\rightarrow$

Log:
$\log: \mathbb{C}^\times \to \mathbb{C}$
$\exp(\log(z)) = z$

derivative of $\frac{\partial}{\partial z}\log(z)$:

Here priciple value : $\log(z) = \log(|z|) + i\arg(z)$
where $-\pi < \arg(z) \leq \pi$



$\log(z)$ is cont in $\Omega \subseteq \mathbb{C} \setminus \mathbb{R}_{\leq 0}$

lemma : let $\Omega_1, \Omega_2 \underset{\text{open}}{\subseteq} \mathbb{C}$

$f : \Omega_1 \to \mathbb{C}$ and
$g : \Omega_2 \to \mathbb{C}$

s.t 1) f, g are continious
2) $f(\Omega_1) \subseteq \Omega_2$
3) $g(f(z)) = z \quad \forall z \in \Omega_1$

If g is diff and $g'(z) \neq 0$, $z \in \Omega_2$ then

$$f'(z) = \frac{1}{g'(f(z))} \quad \forall z \in \Omega_1$$

proof : $a \in \Omega_1$
say $h \in \mathbb{C}$ s.t $|h|$ is small, i.e $a + h \in \Omega_1$

$$g(f(a)) = a$$
and $$g(f(a+h)) = a + h$$

$$1 = \frac{a + h - a}{h}$$

$$= \frac{g(f(a+h)) - g(f(a))}{h}$$

$$= \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \times \frac{f(a+h) - f(a)}{h}$$

now as $h \to 0$

$$\lim_{h \to 0} \left(\frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} \right) \cdot \left(\frac{f(a+h) - f(a)}{h} \right)$$

but f is cont at a

$$\Rightarrow \lim_{h \to 0} f(a+h) - f(a) = 0$$

$$\Rightarrow \lim_{h \to 0} \frac{g(f(a+h)) - g(f(a))}{f(a+h) - f(a)} = g'(f(a)) \neq 0$$

$$\Rightarrow \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$$

$$= \frac{1}{g'(f(a))}$$

(Proof $1 = \frac{a+h-a}{h}$ then substitute values)

$$\Rightarrow f'(a) = \frac{1}{g'(f(a))}$$

Note: putting $f(z) = \log(z)$
$g(z) = \exp(z)$

then $\frac{\partial}{\partial z} \log(z) = \frac{1}{\exp(\log(z))} = \frac{1}{z}$

given log is cont

Recall: If f has a primitive in $\Omega \underset{\text{open}}{\subseteq} \mathbb{C}$ then $\int_\gamma f(z)dz = 0$ $\forall$ closed "basic" γ

but $\int_{|z|=1} \frac{1}{z} dz \neq 0$

even tho $\frac{1}{z} = \frac{\partial}{\partial z} \log(z)$ $z \notin \mathbb{R}_{\gamma, U}$

as $\int_{|z|=1} \frac{1}{z} dz = \int_0^{2\pi} \frac{1}{\cancel{e^{i\theta}}} \cancel{e^{i\theta}}(i)\, d\theta = 2\pi i \neq 0$

$$f(z) = \underset{z = e^{i\theta}}{z}$$

$\gamma(\theta) = e^{i\theta}$
$\gamma'(\theta) = e^{i\theta}(i)$